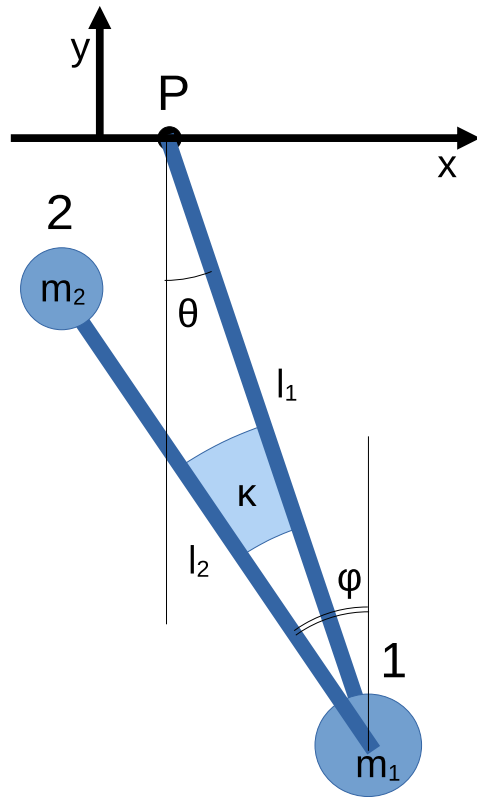


The analytical equations of the Pendulum-Inverted Pendulum oscillator



The Cartesian coordinates of the two masses are

$$\begin{cases} x_1 = l_1 \sin(\theta) + x_P(t) \\ y_1 = -l_1 \cos(\theta) \\ x_2 = x_1 - l_2 \sin(\phi) \\ y_2 = y_1 + l_2 \cos(\phi) \end{cases} \quad \Rightarrow \quad \begin{cases} x_1 = l_1 \sin(\theta) + x_P(t) \\ y_1 = -l_1 \cos(\theta) \\ x_2 = l_1 \sin(\theta) - l_2 \sin(\phi) + x_P(t) \\ y_2 = -l_1 \cos(\theta) + l_2 \cos(\phi) \end{cases}$$

An elastic potential energy $U_e = \frac{1}{2} \kappa (\theta - \phi)^2$ is stored in the spring.

Based on the second order development of x_1 , y_1 , x_2 , y_2 an approximate Lagrangian can be written in terms of θ and ϕ .

More convenient variables can then be introduced such as

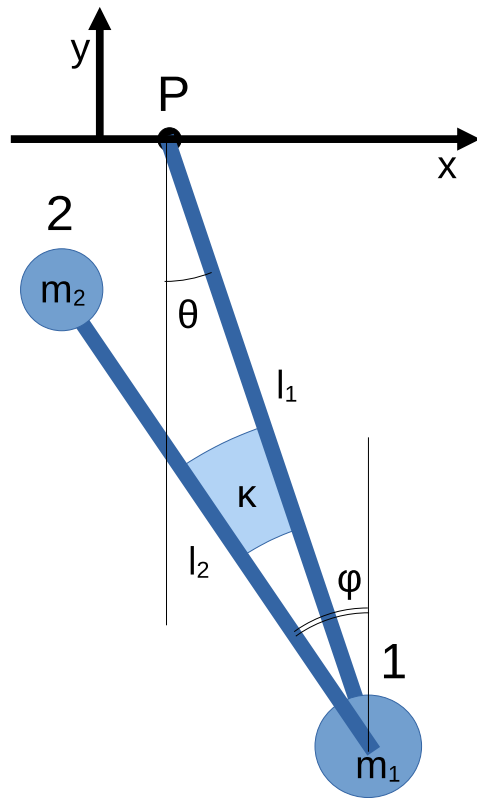
$$\begin{cases} x = l_1 \theta - l_2 \phi \\ z = l_2 \theta + l_1 \phi \end{cases}$$

Piero Chessa

BHETSA Meeting
Pisa, 11/11/2023

(revised)

The analytical equations of the Pendulum-Inverted Pendulum oscillator



In terms of the variables
$$\begin{cases} x = l_1 \theta - l_2 \phi \\ z = l_2 \theta + l_1 \phi \end{cases}$$

the Lagrange's equations of the system can be summarized as

$$\mathbf{M} \begin{pmatrix} \ddot{x} \\ \ddot{z} \end{pmatrix} = -\frac{g}{l_2} \mathbf{K} \begin{pmatrix} x \\ z \end{pmatrix} - \mathbf{N} \ddot{x}_P$$

where the constant matrices \mathbf{M} , \mathbf{K} and \mathbf{N} are, respectively,

$$\mathbf{M} = \begin{pmatrix} \mu \lambda^4 + (\lambda^2 + 1)^2 & \mu \lambda^3 \\ \mu \lambda^3 & \mu \lambda^2 \end{pmatrix}$$

$$\mathbf{K} = \begin{pmatrix} (\mu + 1) \lambda^3 - 1 + \gamma (\lambda^2 + 1)^2 & (\mu + 1) \lambda^2 + \lambda - \gamma (\lambda^2 - 1) \\ (\mu + 1) \lambda^2 + \lambda - \gamma (\lambda^2 - 1) & (\mu + 1) \lambda - \lambda^2 + \gamma (\lambda - 1)^2 \end{pmatrix}$$

$$\mathbf{N} = (\lambda^2 + 1) \begin{pmatrix} (\mu + 1) \lambda^2 + 1 \\ \mu \lambda \end{pmatrix}$$

and some dimensionless parameters have been introduced:

$$\mu = \frac{m_1}{m_2}, \quad \lambda = \frac{l_1}{l_2}, \quad \gamma = \frac{\kappa}{m_2 g l_2}.$$

Oscillations

The equation describes a forced 2-DoF oscillator. Here we focus on the free oscillations (although the complete transfer function is available for further study).

$$\mathbf{M} \begin{pmatrix} \ddot{x} \\ \ddot{z} \end{pmatrix} = -\frac{g}{l_2} \mathbf{K} \begin{pmatrix} x \\ z \end{pmatrix} - \cancel{\mathbf{N} \ddot{x}_p} \quad \Rightarrow \quad \begin{pmatrix} \ddot{x} \\ \ddot{z} \end{pmatrix} = -\frac{g}{l_2} \mathbf{M}^{-1} \mathbf{K} \begin{pmatrix} x \\ z \end{pmatrix}$$

The Laplace transform of the equation gives

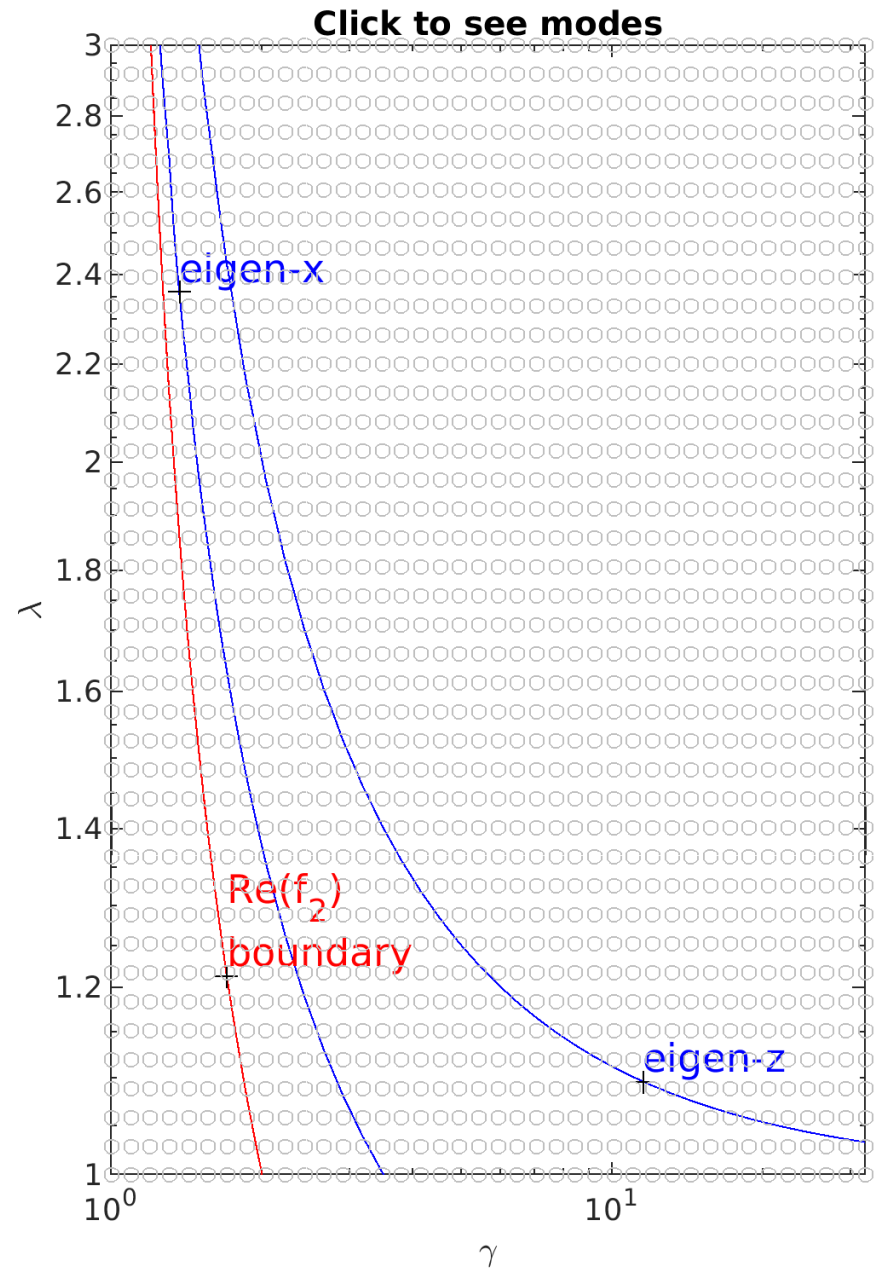
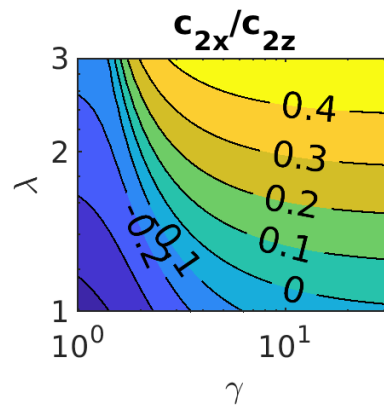
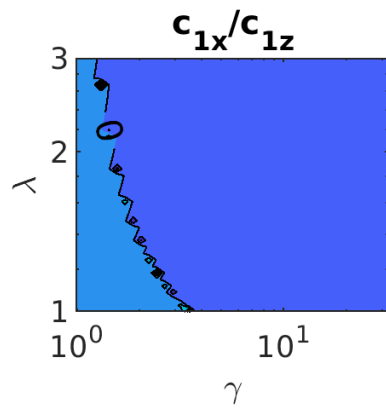
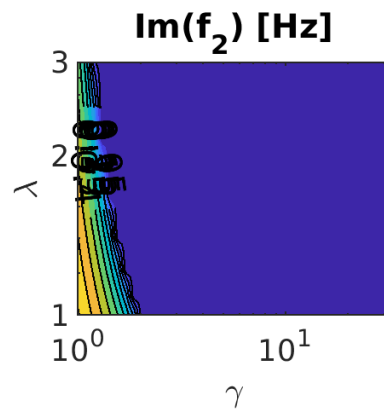
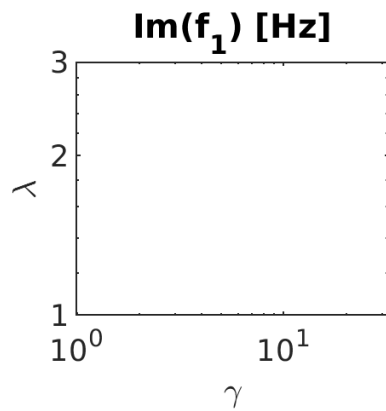
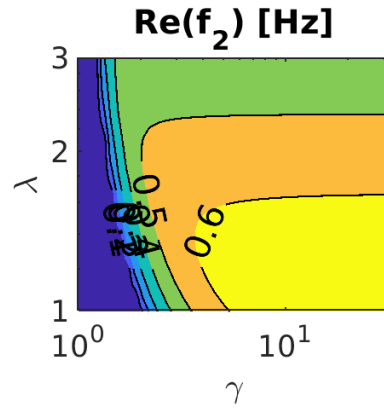
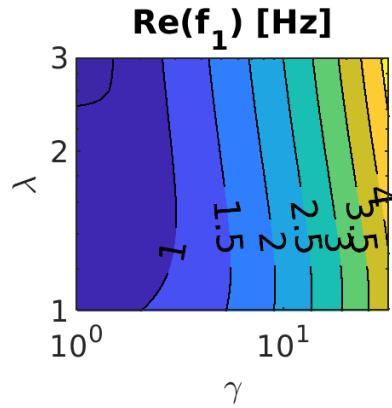
$$s^2 \begin{pmatrix} \tilde{x} \\ \tilde{z} \end{pmatrix} = -\frac{g}{l_2} \mathbf{M}^{-1} \mathbf{K} \begin{pmatrix} \tilde{x} \\ \tilde{z} \end{pmatrix} \quad \text{or} \quad \left(s^2 + \frac{g}{l_2} \mathbf{M}^{-1} \mathbf{K} \right) \begin{pmatrix} \tilde{x} \\ \tilde{z} \end{pmatrix} = 0.$$

By putting the $\mathbf{M}^{-1} \mathbf{K}$ matrix in the diagonal form, one finds the natural frequencies of the system and the eigen-states of the oscillator that are presented hereafter.

All plots are worked out with $l_2 = 0.5\text{m}$. All **frequencies scale like** $\frac{1}{\sqrt{l_2}}$.

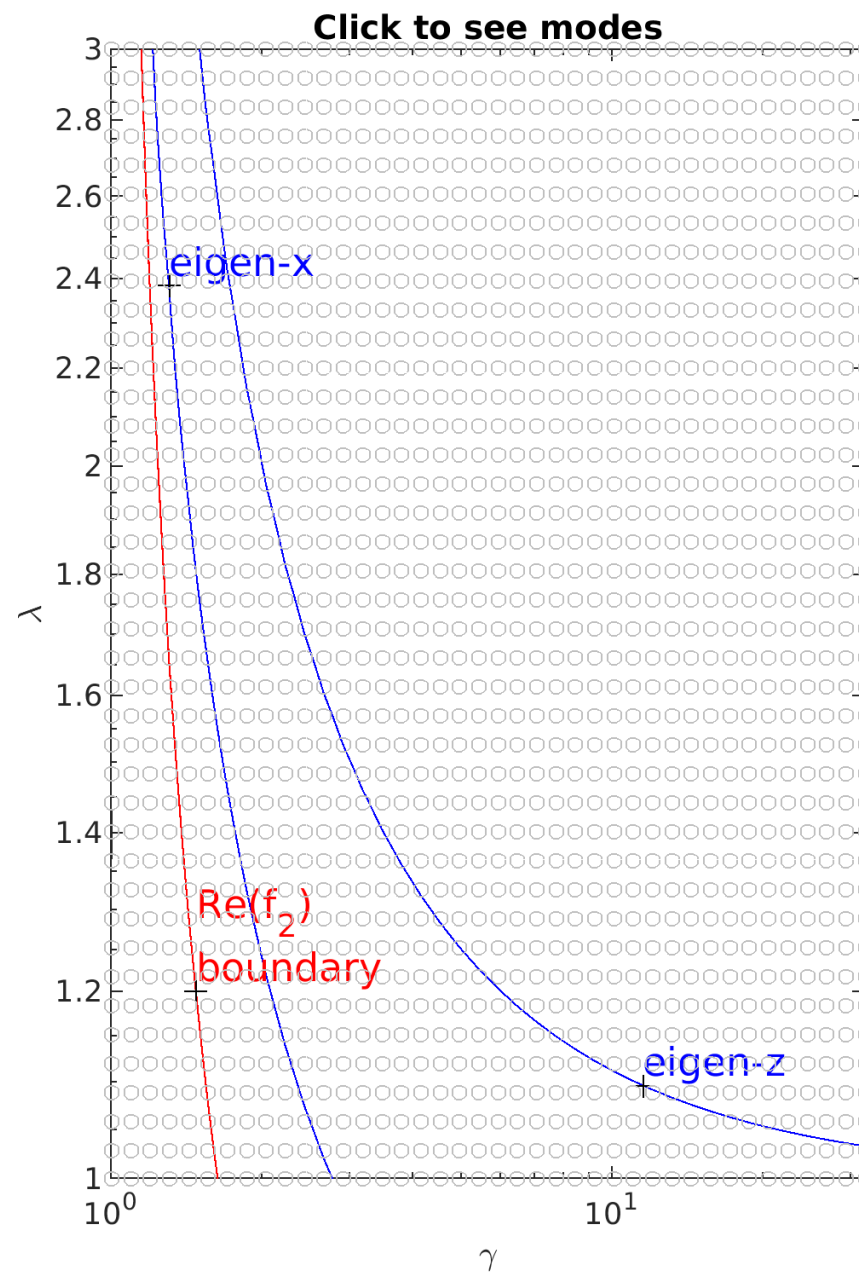
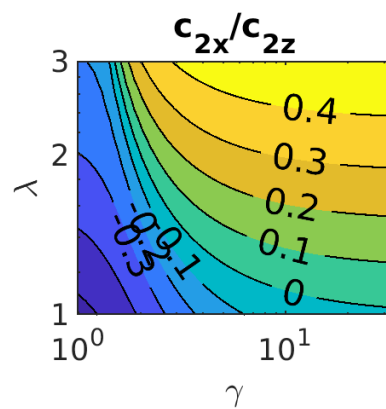
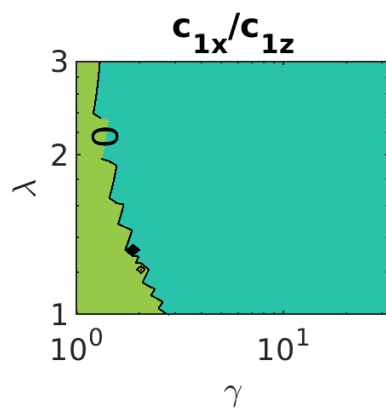
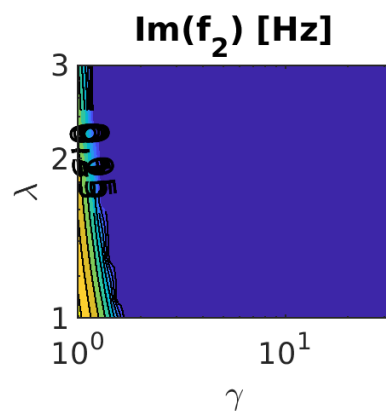
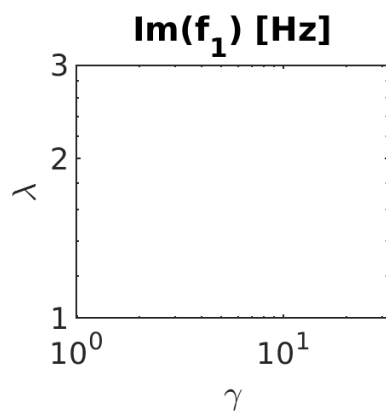
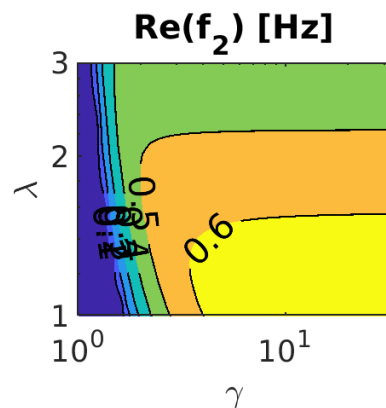
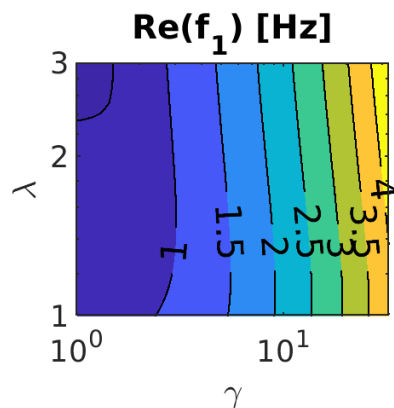
$$\mu = 1$$

$$[l_2 = 0.5m, m_1 = m_2]$$



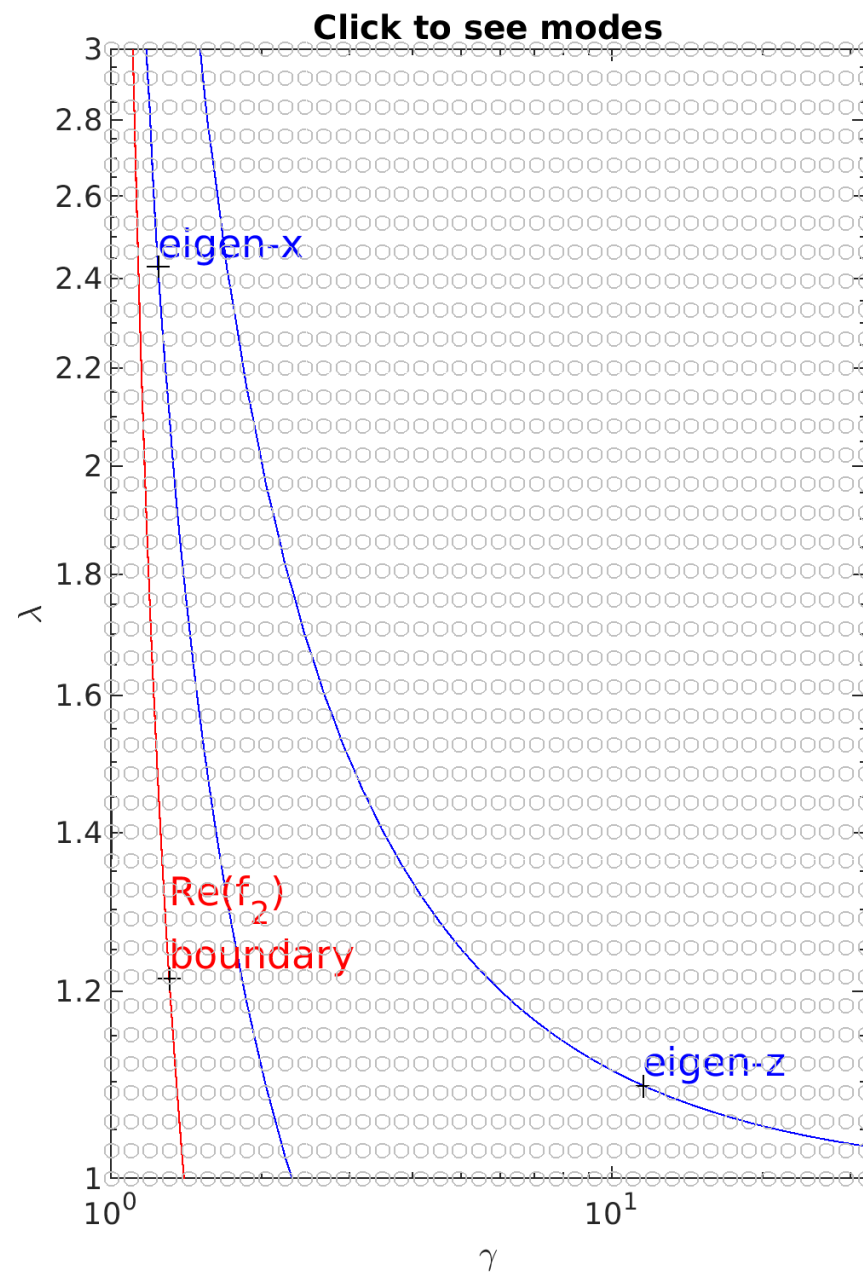
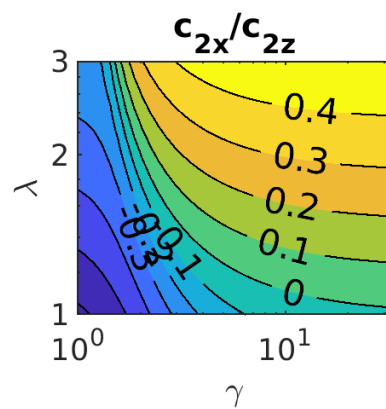
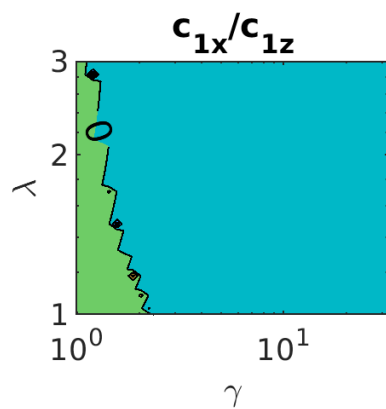
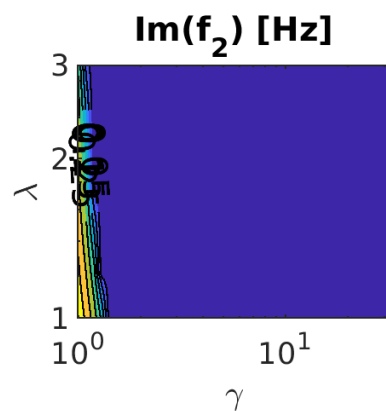
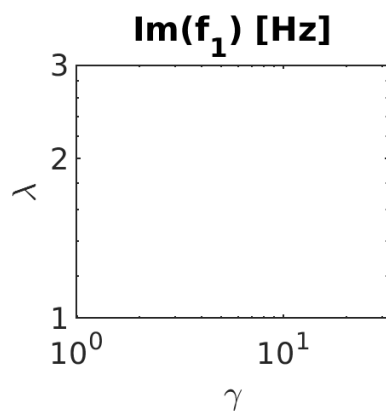
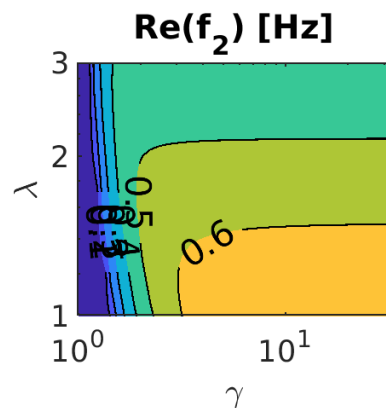
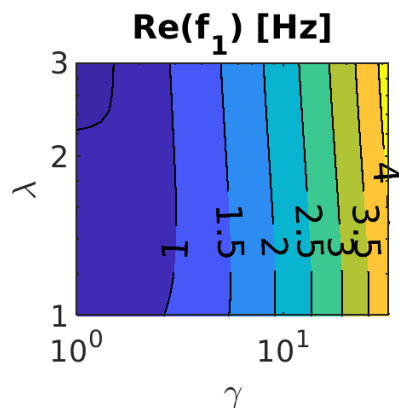
$$\mu = 1.59$$

$$[I_2 = 0.5m, m_1 = 1.59 m_2]$$



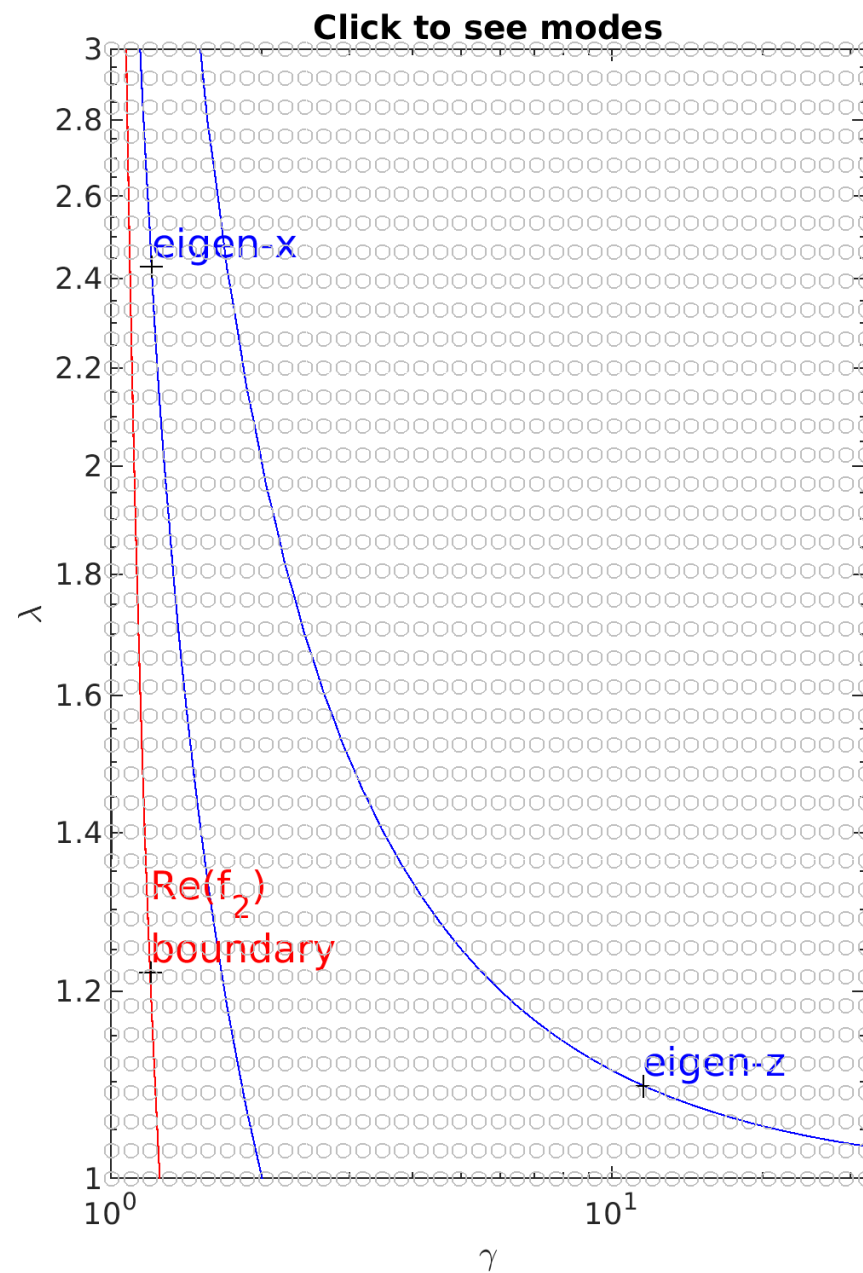
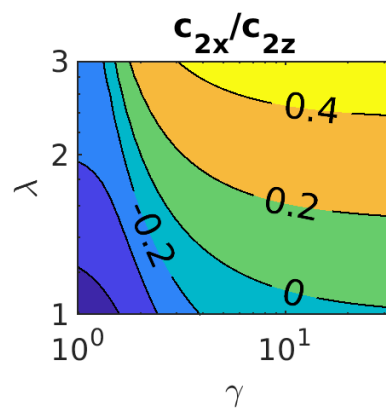
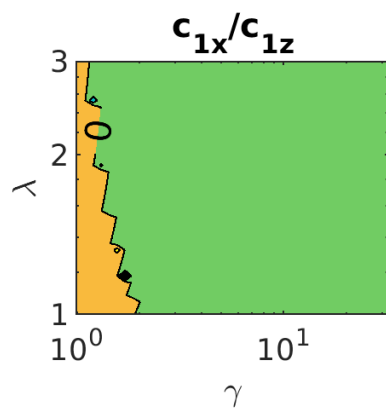
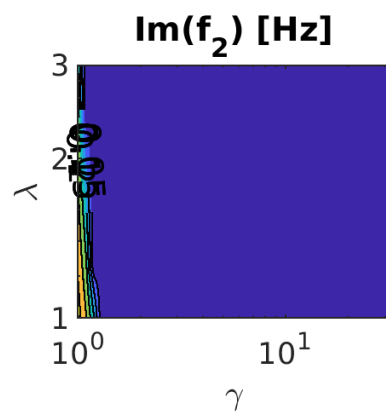
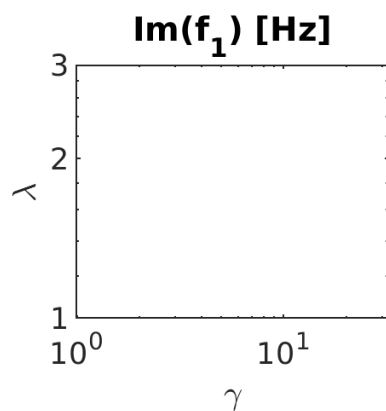
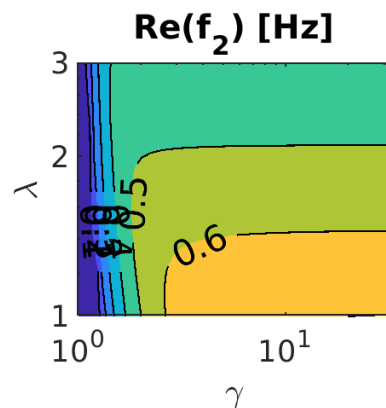
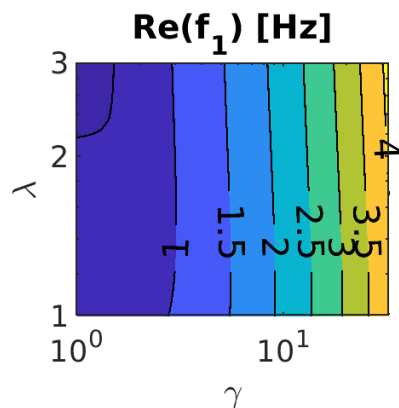
$$\mu = 2.52$$

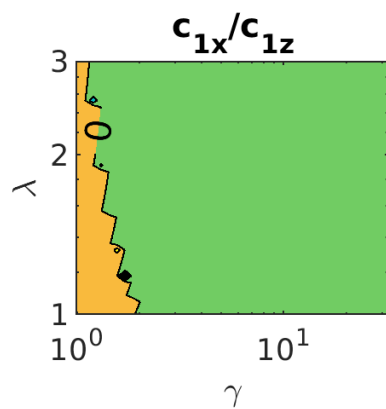
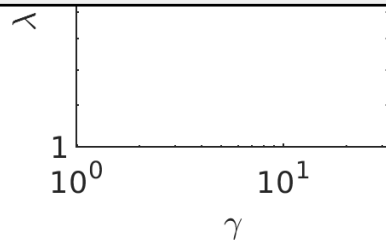
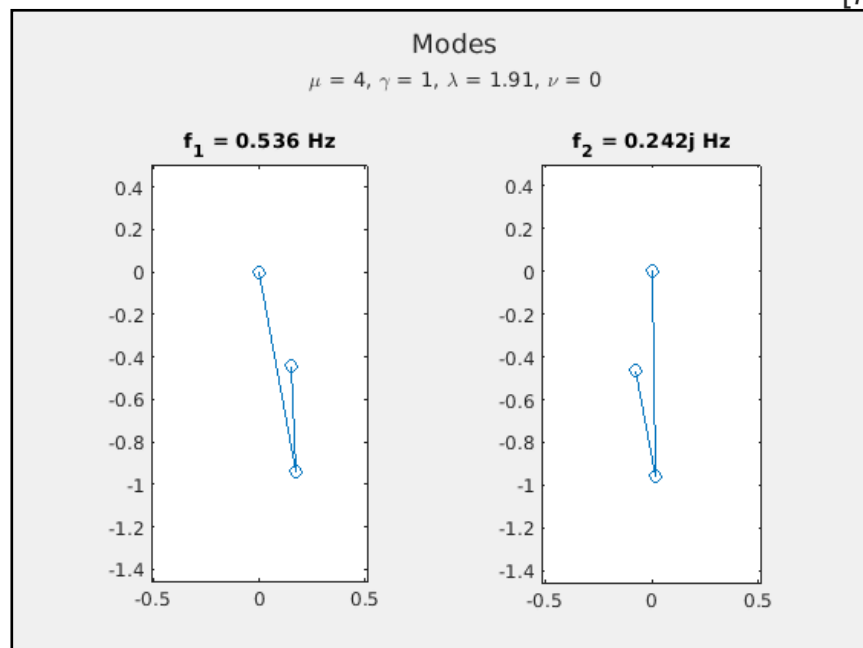
$$[I_2 = 0.5m, m_1 = 2.52 m_2]$$



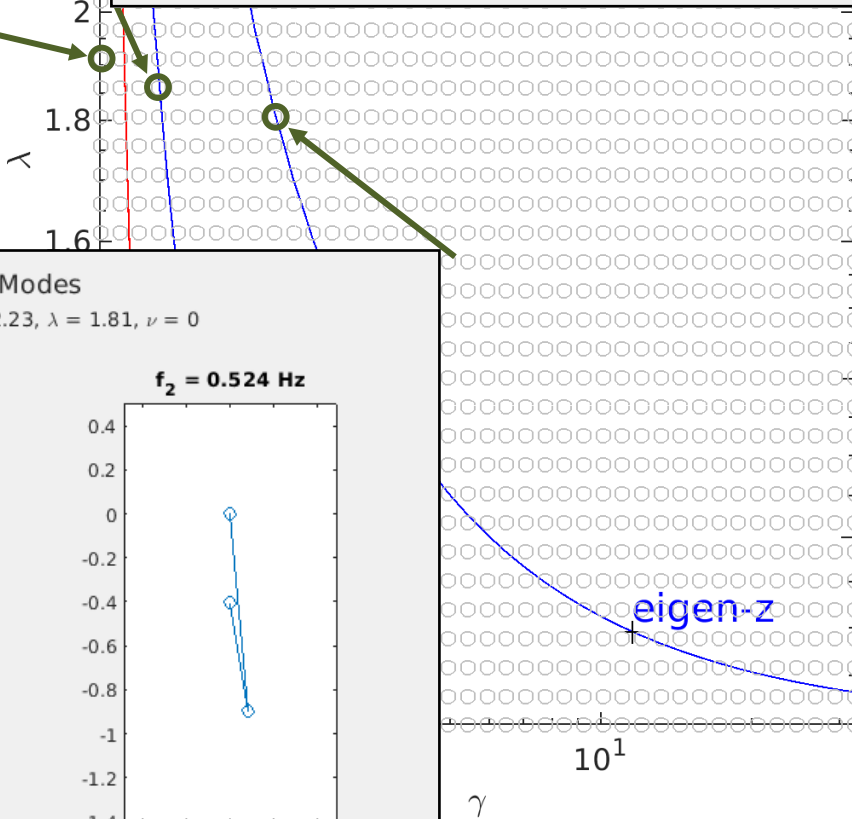
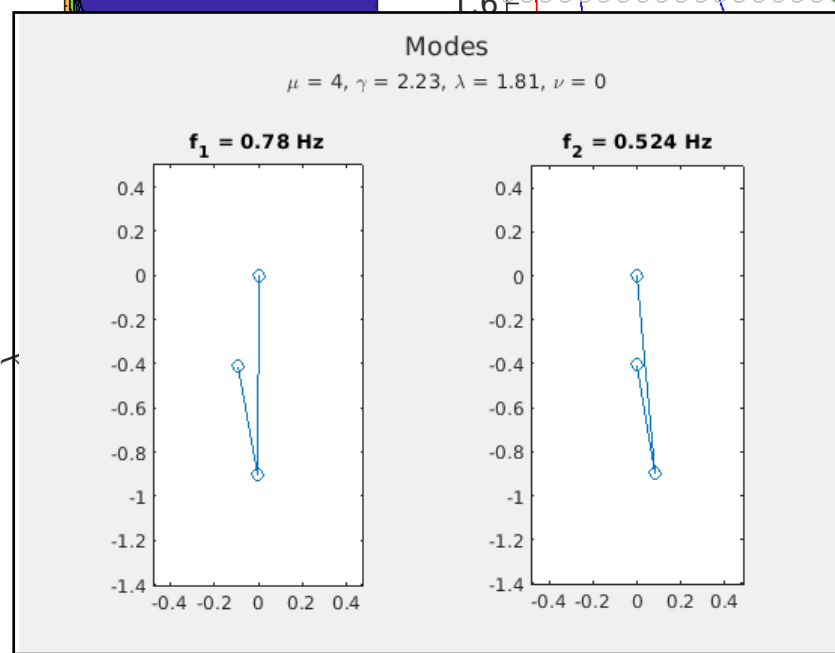
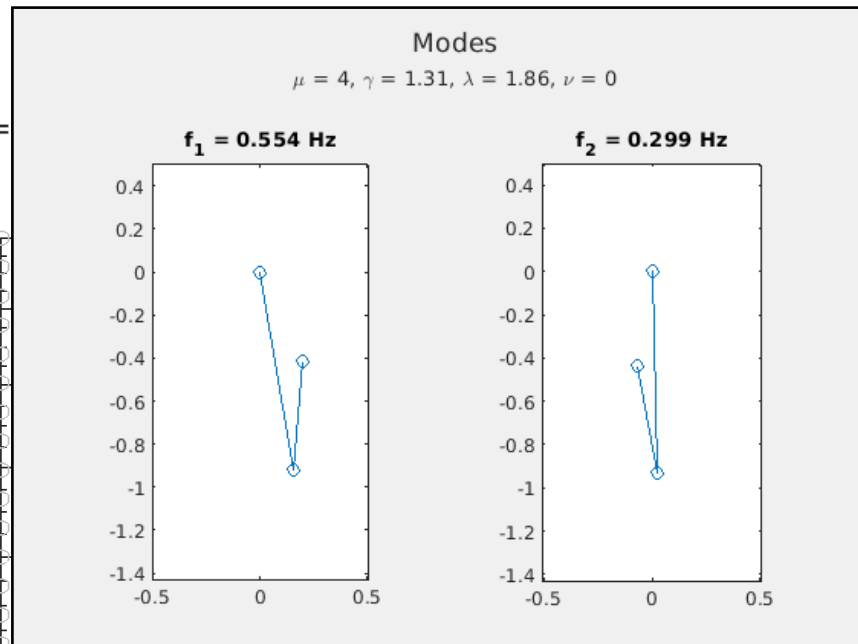
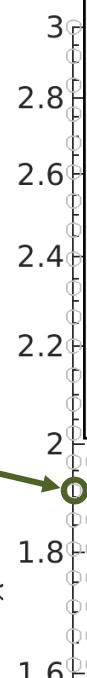
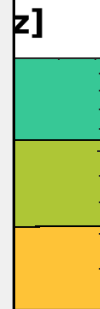
$$\mu = 4$$

$$[l_2 = 0.5m, m_1 = 4 m_2]$$

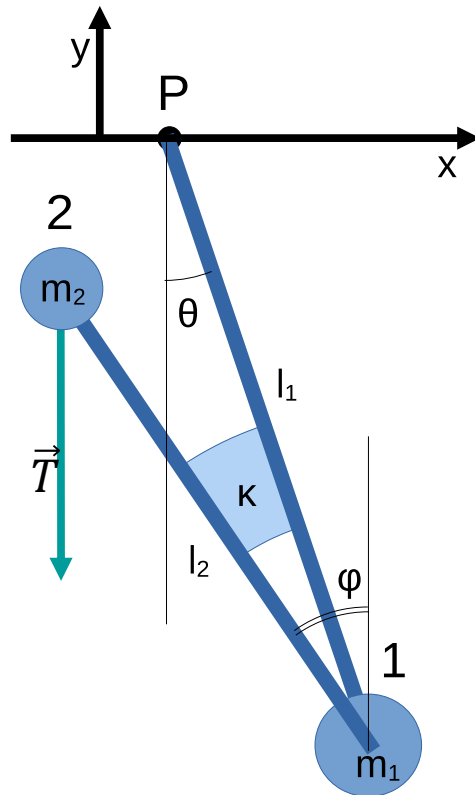




$\mu = 4$
 $[l_2 = 0.5m, m_1 =$



The Pendulum-Inverted Pendulum oscillator with a constant vertical load



Let's suppose that a load is suspended to the device and that this load can be approximated with a constant vertical force.

The Lagrange's equations of the system become

$$\mathbf{M} \begin{pmatrix} \ddot{x} \\ \ddot{z} \end{pmatrix} = -\frac{g}{l_2} (\mathbf{K} + \mathbf{K}_T) \begin{pmatrix} x \\ z \end{pmatrix} - \mathbf{N} \ddot{x}_P$$

where a new constant matrix \mathbf{K}_T is introduced

$$\mathbf{K}_T = v \begin{pmatrix} \lambda^3 - 1 & \lambda(\lambda + 1) \\ \lambda(\lambda + 1) & \lambda(1 - \lambda) \end{pmatrix}$$

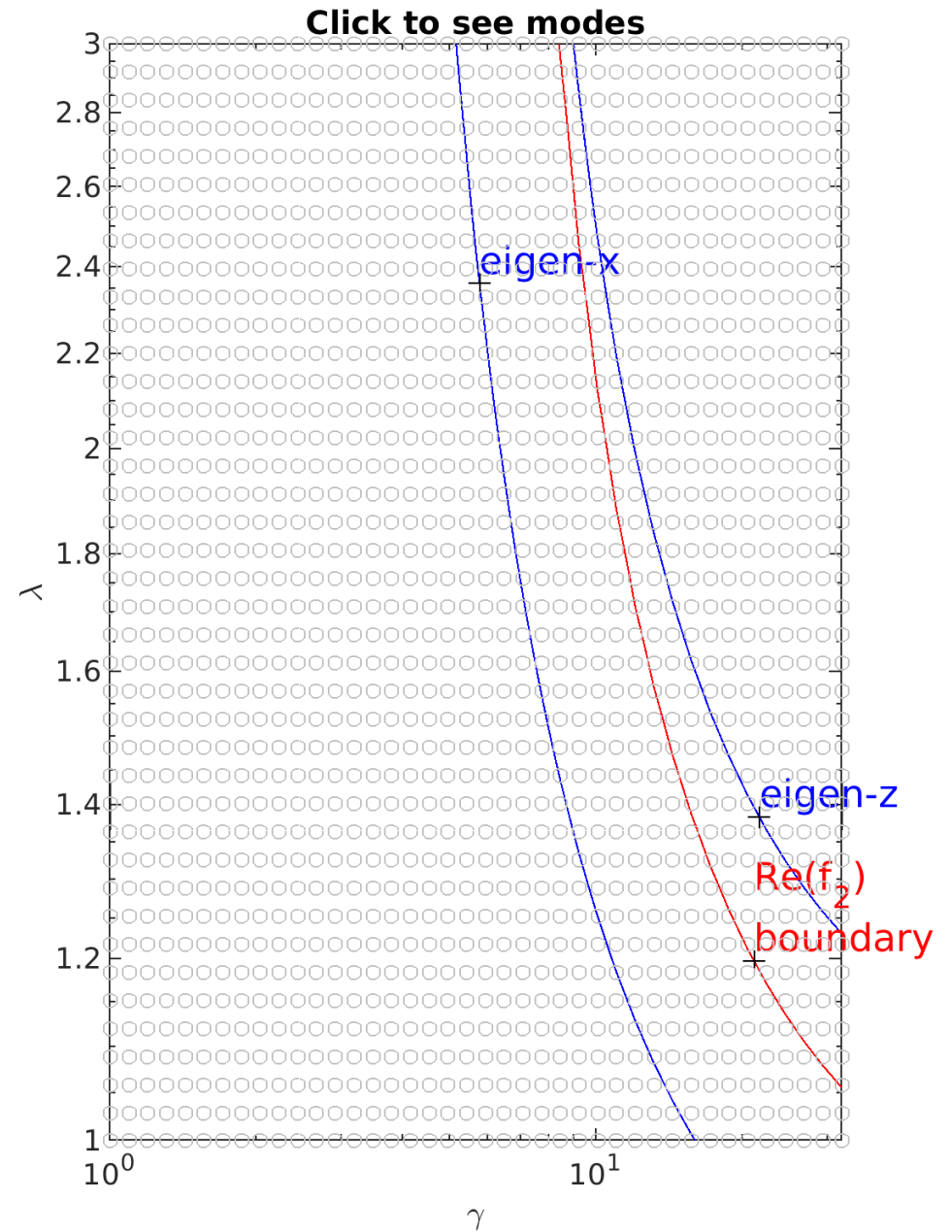
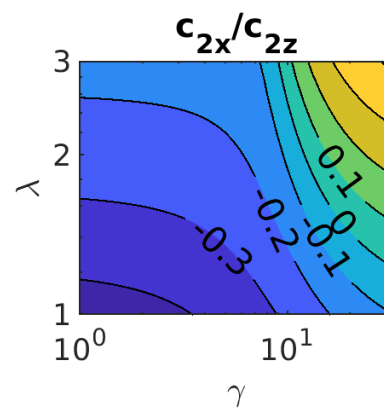
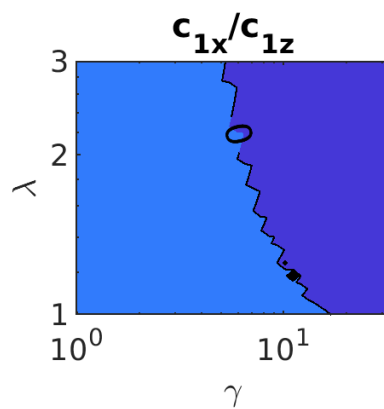
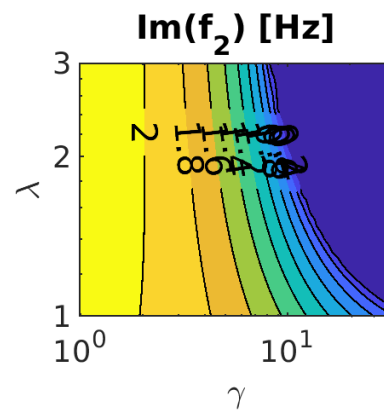
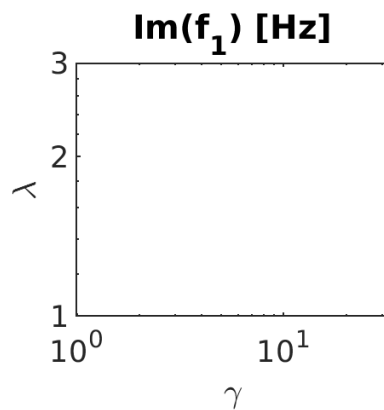
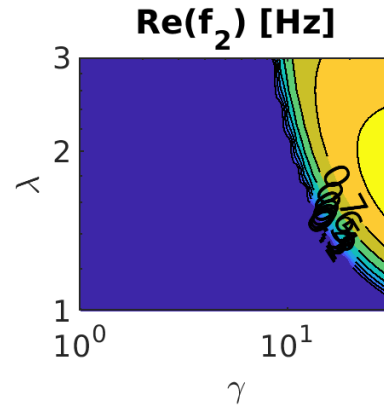
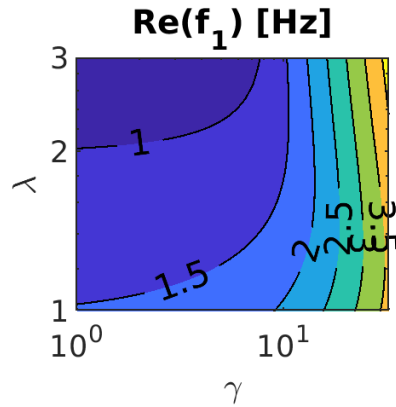
along with the new dimensionless parameter $v = \frac{T}{m_2 g}$.

The natural modes of the loaded system can thus be found by solving

$$\left[s^2 + \frac{g}{l_2} \mathbf{M}^{-1} (\mathbf{K} + \mathbf{K}_T) \right] \begin{pmatrix} \tilde{x} \\ \tilde{z} \end{pmatrix} = 0$$

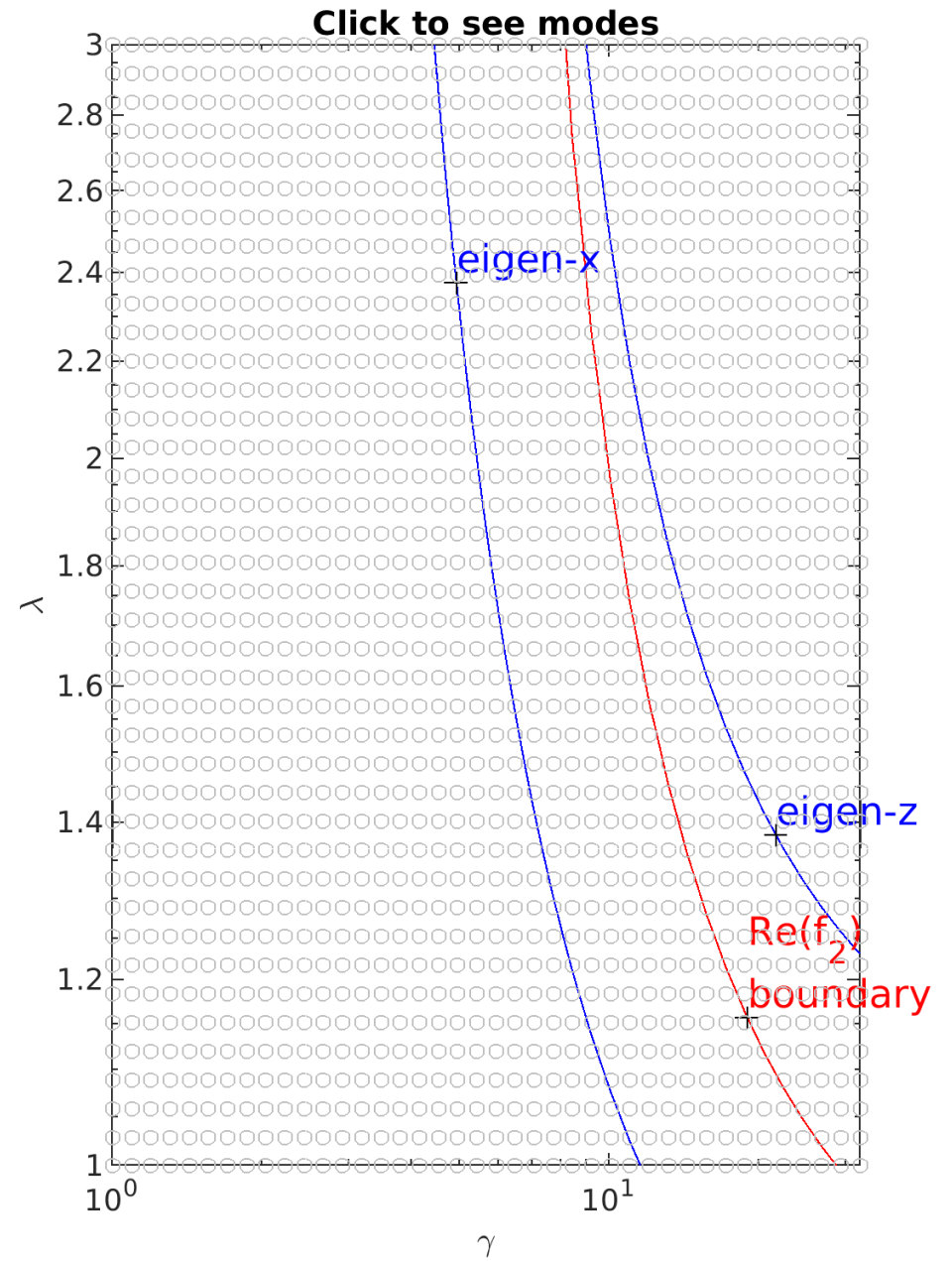
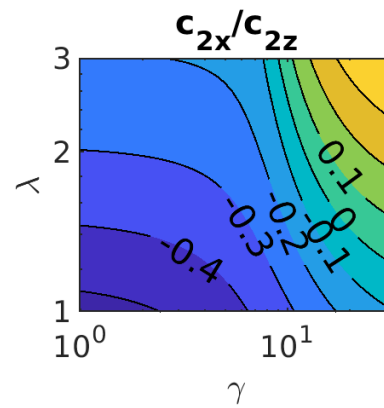
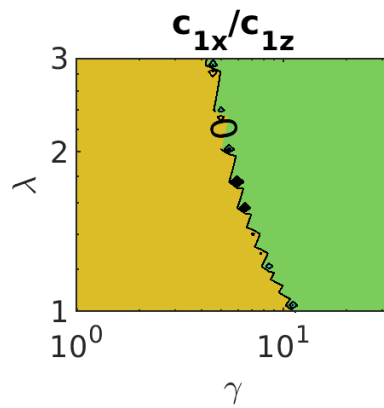
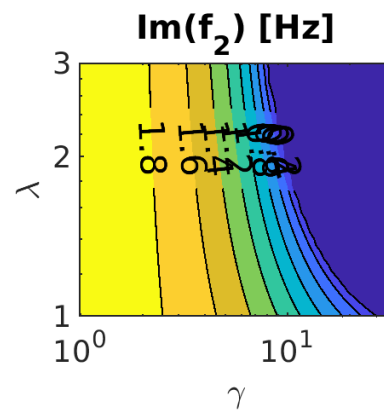
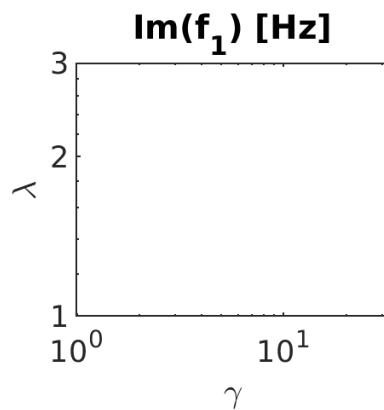
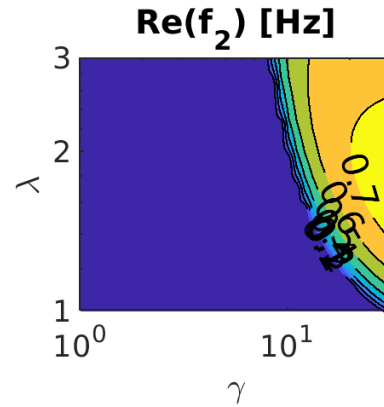
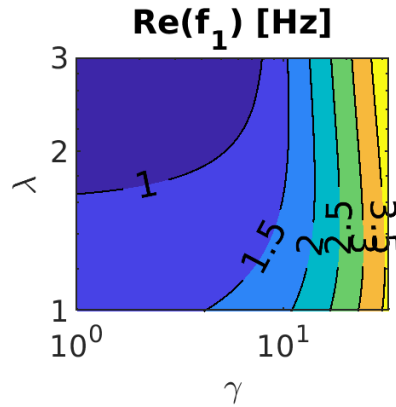
$$\mu = 1, \nu = 5$$

$$[l_2 = 0.5m, m_1 = m_2, T = 5 m_2 g]$$



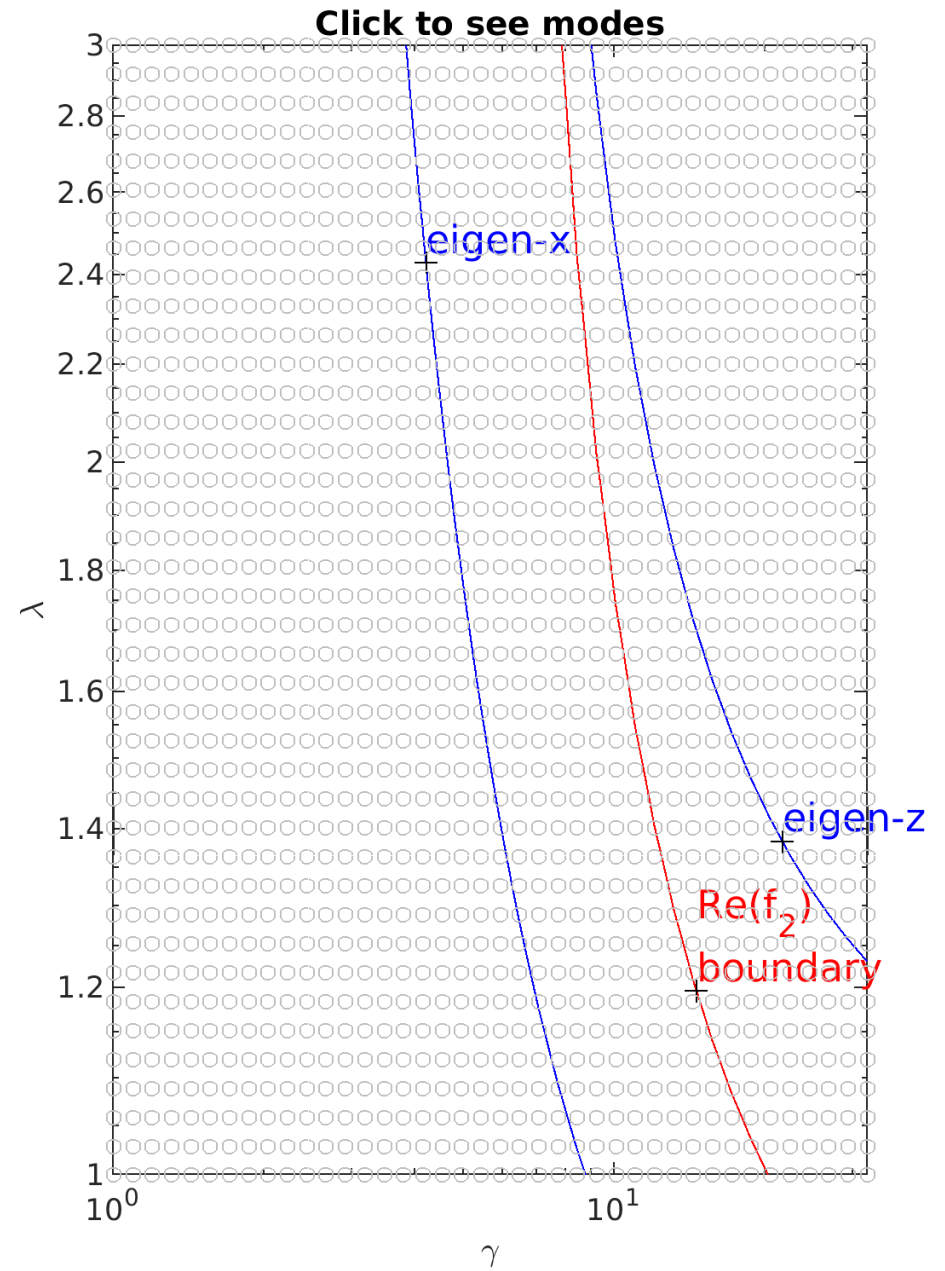
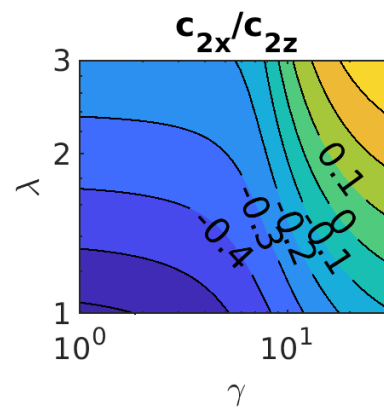
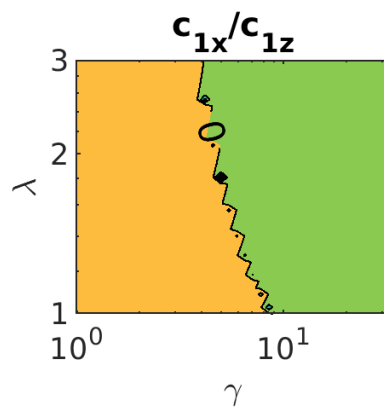
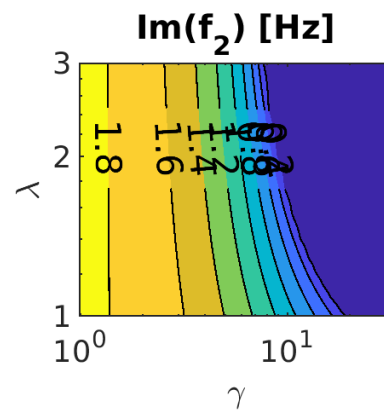
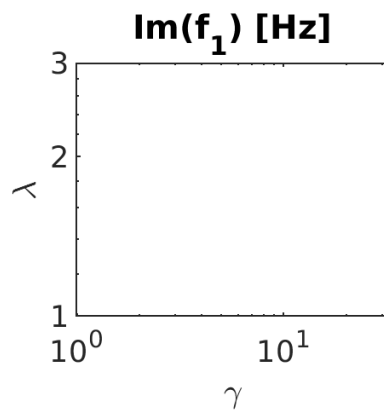
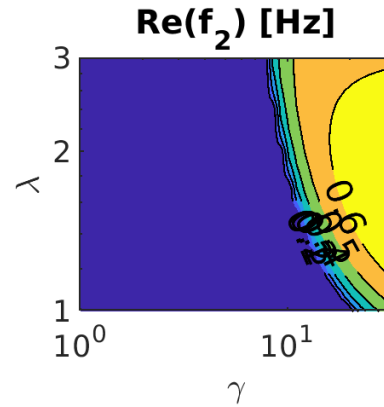
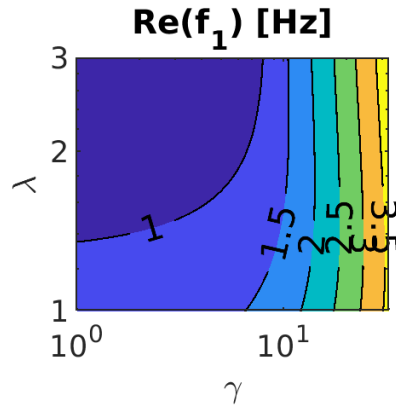
$$\mu = 1.59, \nu = 5$$

$$[l_2 = 0.5m, m_1 = 1.59 m_2, T = 5 m_2 g]$$



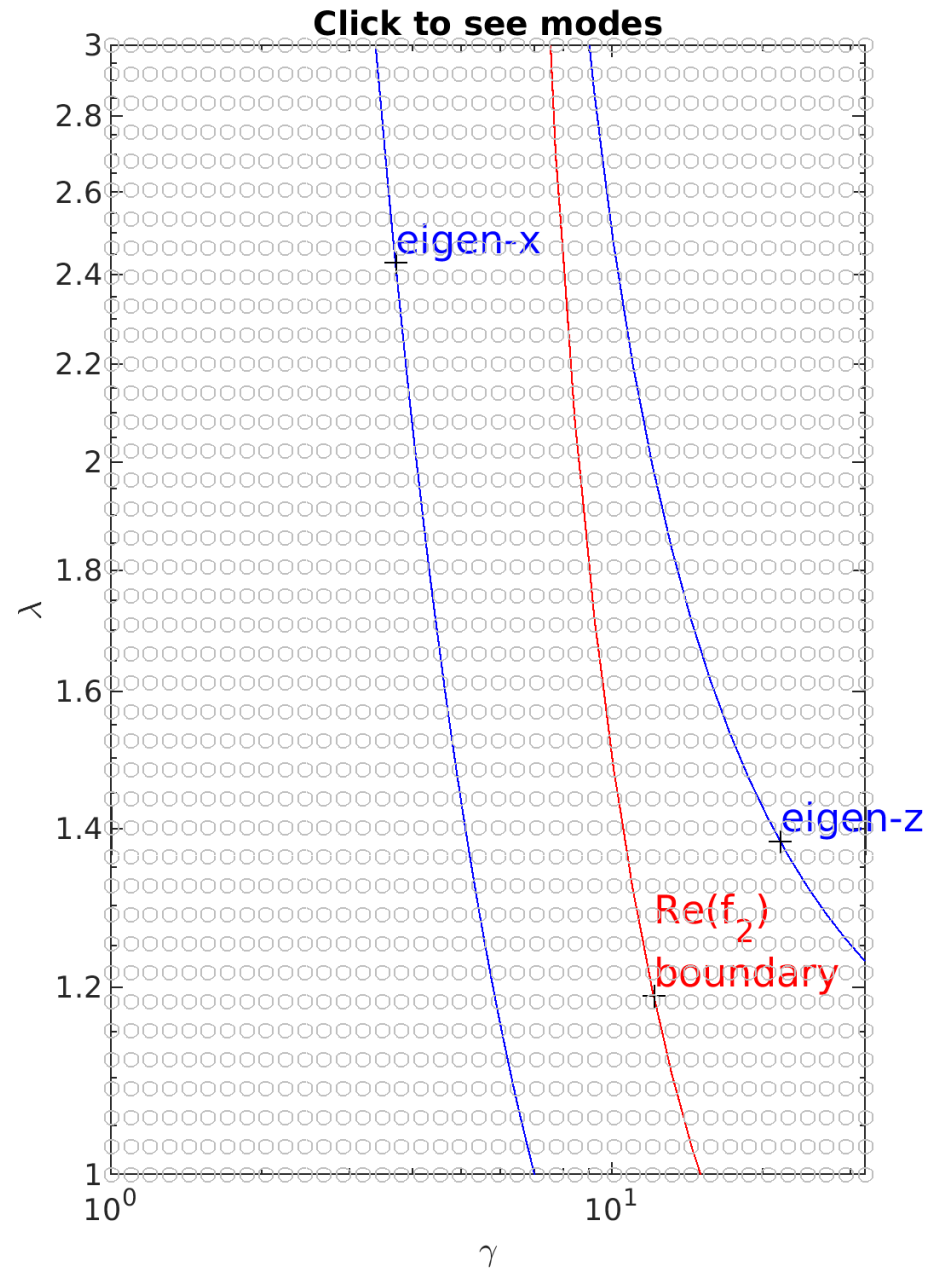
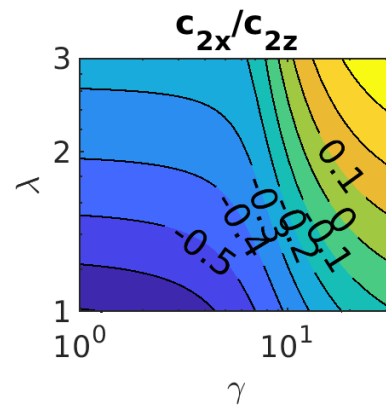
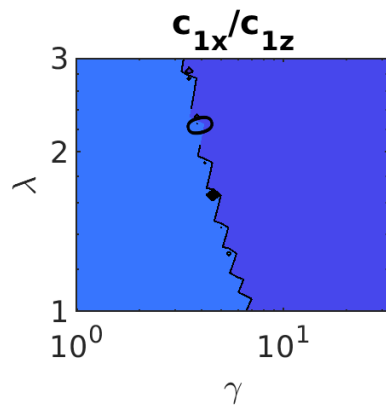
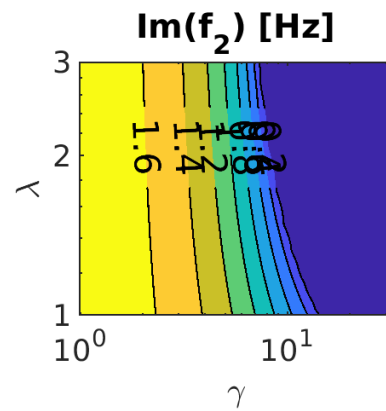
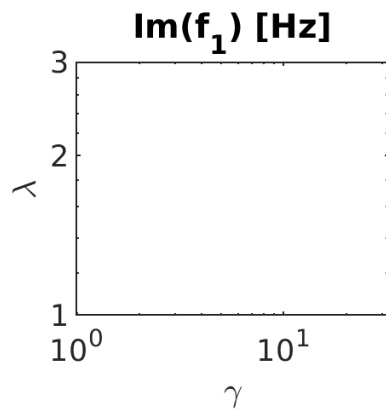
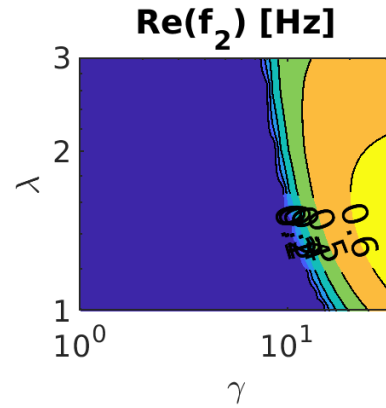
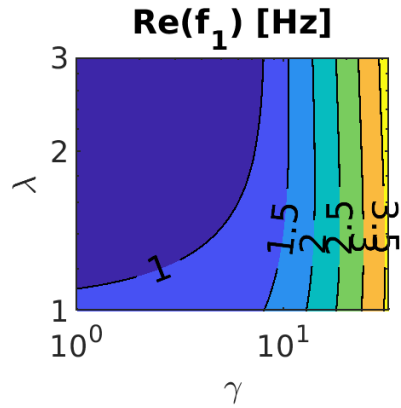
$$\mu = 2.52, \nu = 5$$

$$[I_2 = 0.5m, m_1 = 2.52 m_2, T = 5 m_2 g]$$



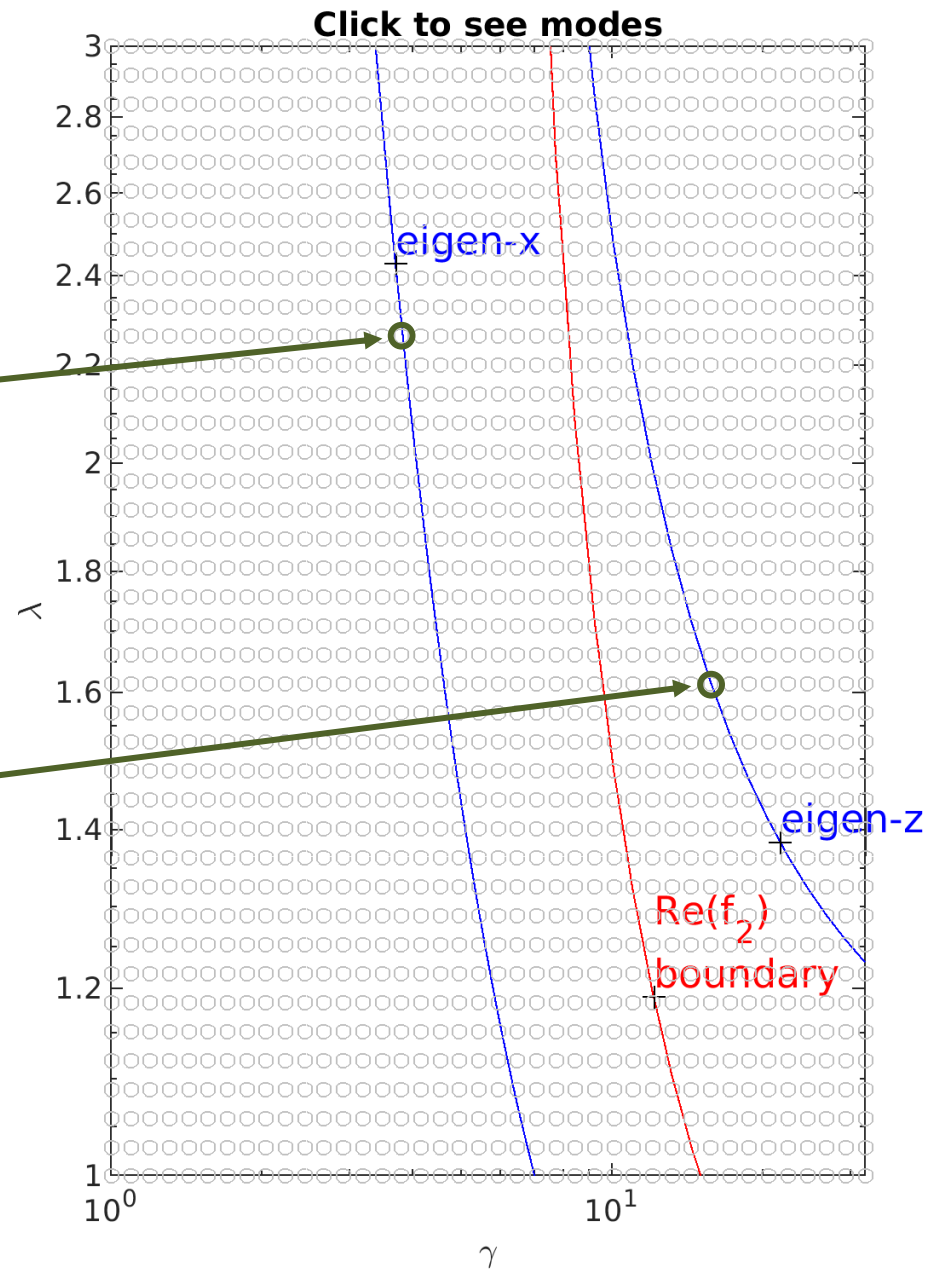
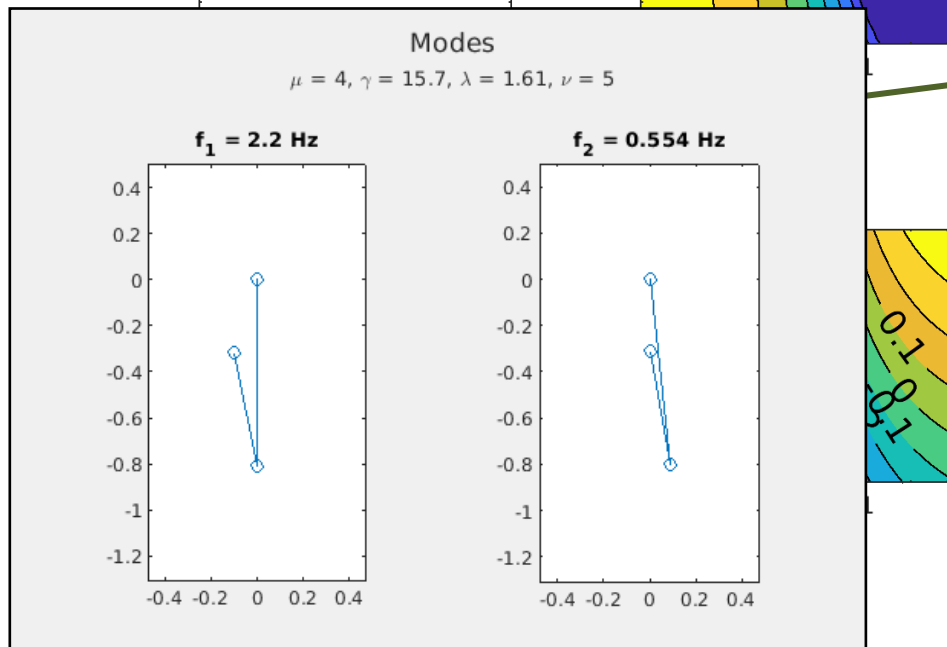
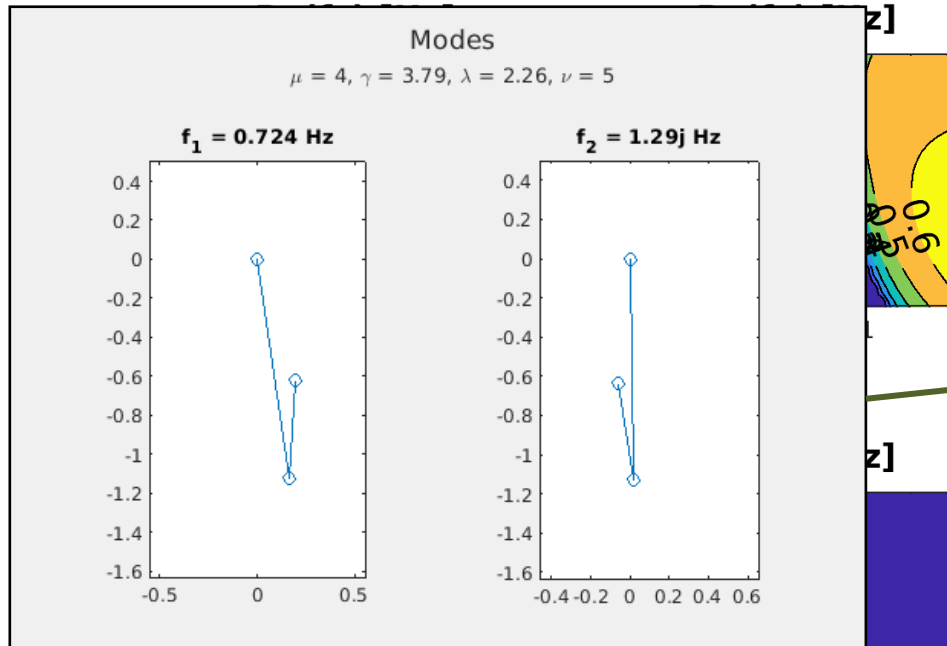
$$\mu = 4, \nu = 5$$

$$[l_2 = 0.5m, m_1 = 4 m_2, T = 5 m_2 g]$$



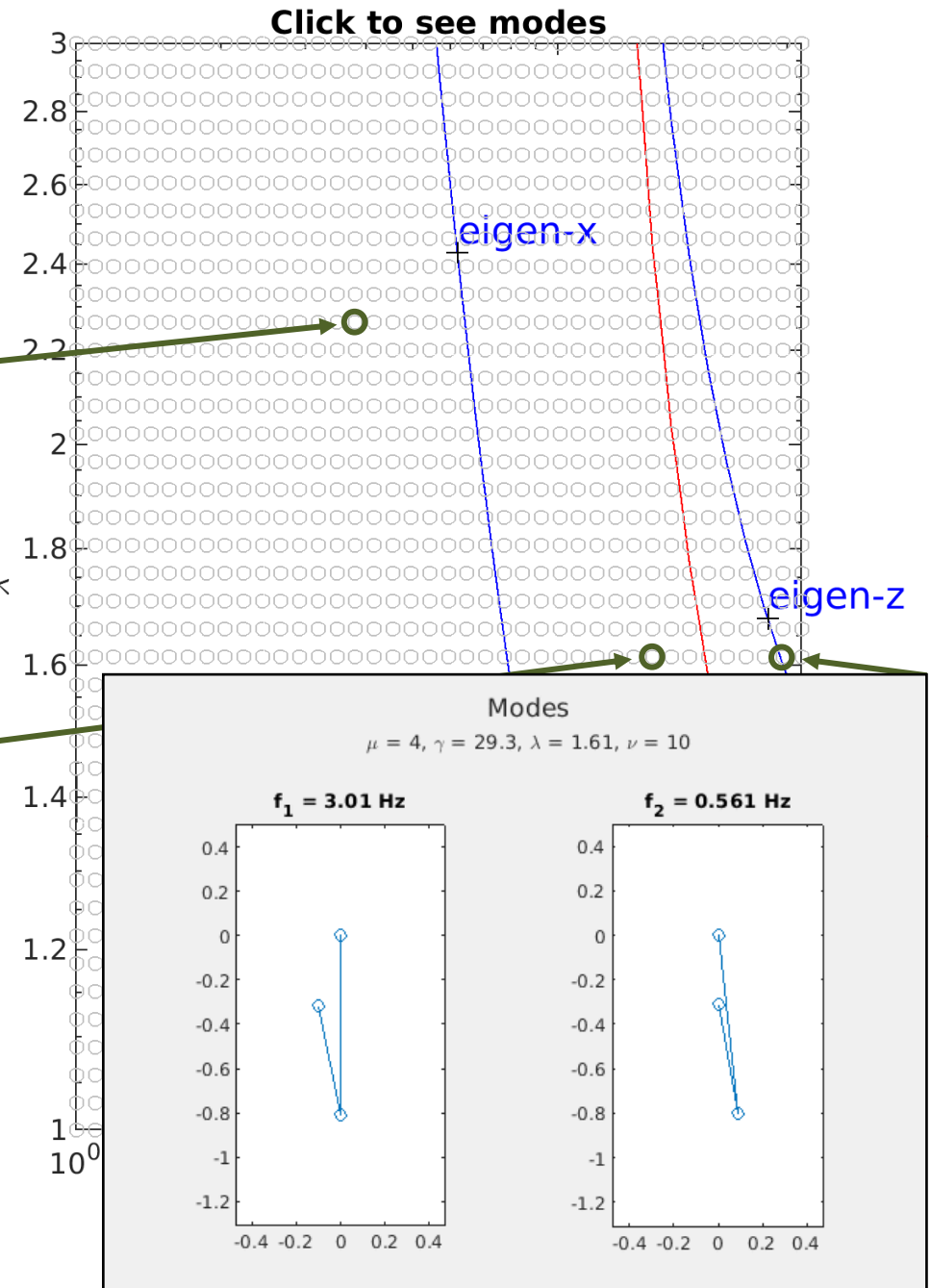
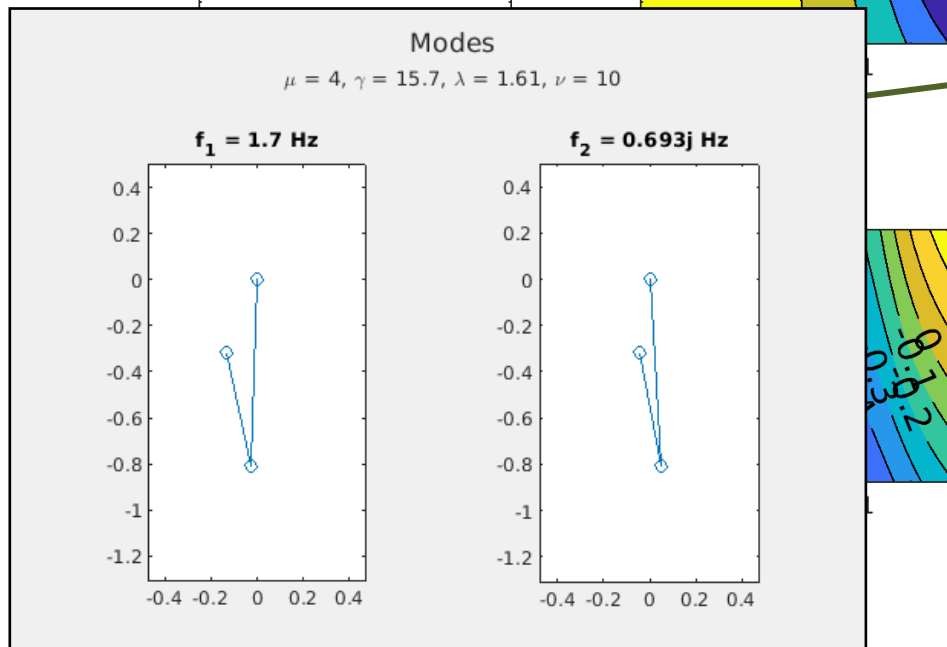
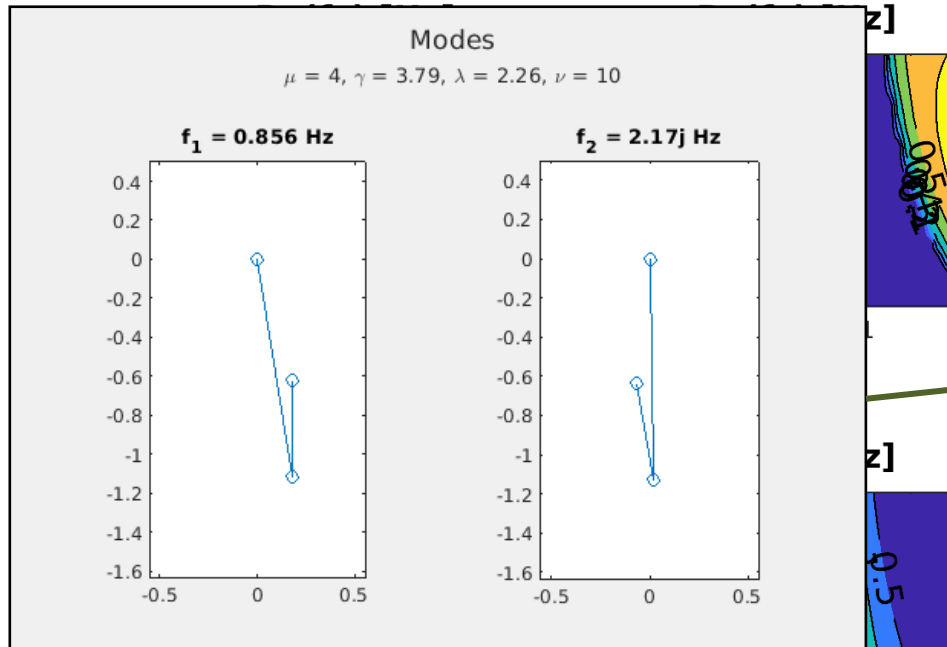
$$\mu = 4, \nu = 5$$

$$[l_2 = 0.5m, m_1 = 4 m_2, T = 5 m_2 g]$$



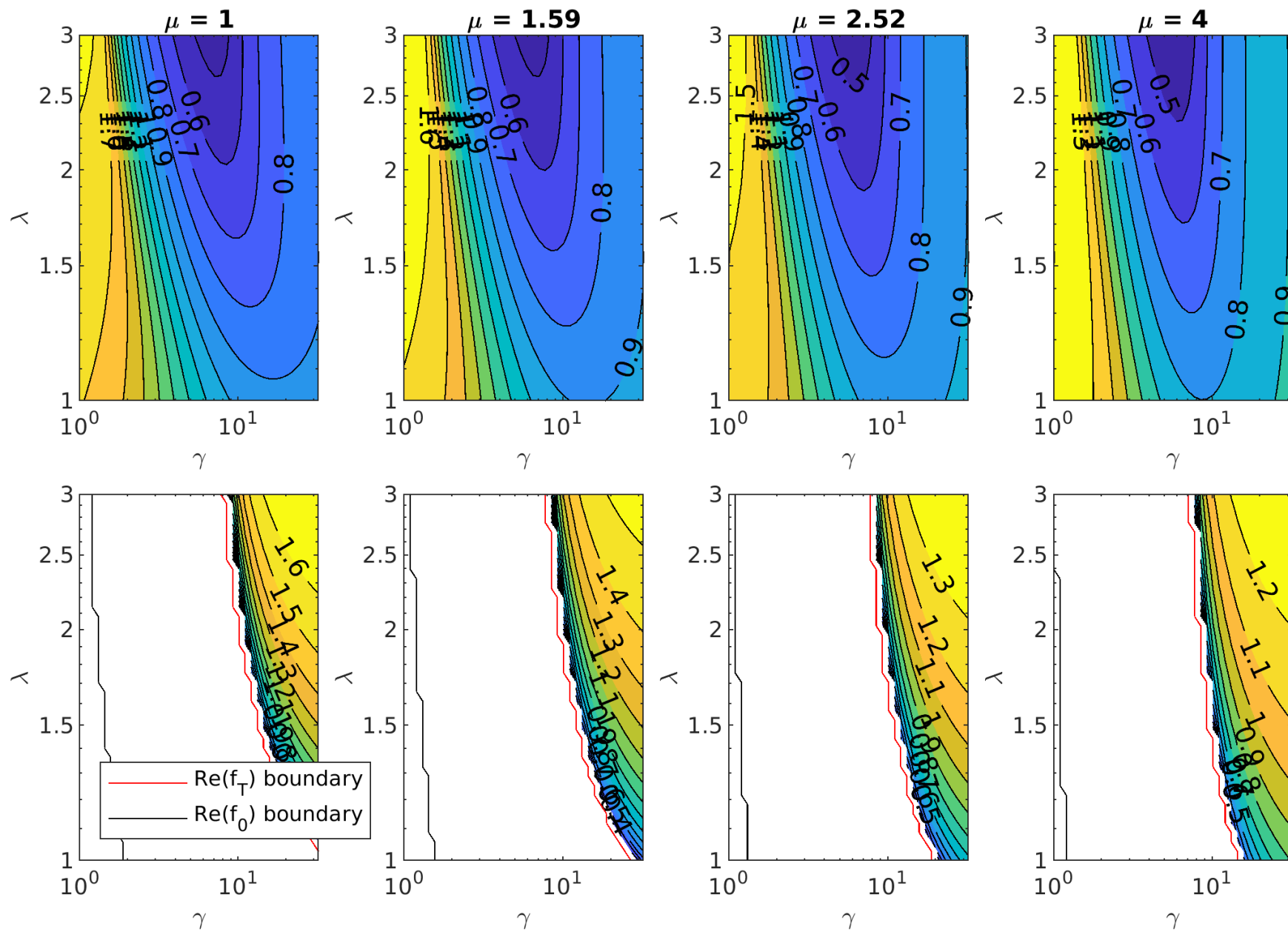
$$\mu = 4, \nu = 10$$

$$[I_2 = 0.5m, m_1 = 4 m_2, T = 10 m_2 g]$$



$$\frac{f_T}{f_0}$$

$$T = 5 \cdot m_2 g$$



Variable definitions and values

System variables:
$$\begin{cases} x = l_1 \theta - l_2 \phi \\ z = l_2 \theta + l_1 \phi \end{cases}$$

Parameter definitions:	$\mu = \frac{m_1}{m_2}$	$\lambda = \frac{l_1}{l_2}$	$\gamma = \frac{\kappa}{m_2 g l_2}$	$\nu = \frac{T}{m_2 g}$
Interval:	1 to 4	1 to 3	1 to 32	0, 5, 10
No of values:	4	64	64	3
Equally spaced in:	Log scale	Log scale	Log scale	Lin scale

All plots are worked out with $l_2 = 0.5\text{m}$. All frequencies scale like $\frac{1}{\sqrt{l_2}}$.