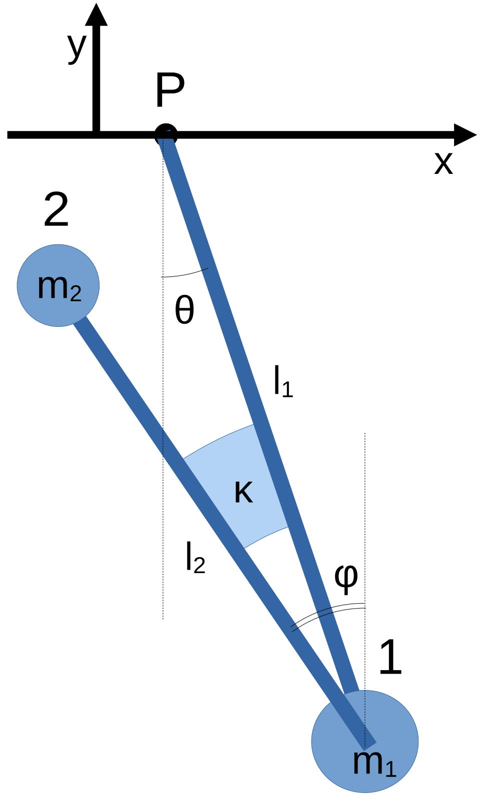
The analytical equations of the Pendulum-Inverted Pendulum oscillator



The Cartesian coordinates of the two masses are

$$\begin{cases} x_1 = l_1 \sin(\theta) + x_P(t) \\ y_1 = -l_1 \cos(\theta) \\ \\ x_2 = x_1 - l_2 \sin(\phi) \\ y_2 = y_1 + l_2 \cos(\phi) \end{cases} \Rightarrow \begin{cases} x_1 = l_1 \sin(\theta) + x_P(t) \\ y_1 = -l_1 \cos(\theta) \\ x_2 = l_1 \sin(\theta) - l_2 \sin(\phi) + x_P(t) \\ y_2 = -l_1 \cos(\theta) + l_2 \cos(\phi) \end{cases}$$

An elastic potential energy $U_e = \frac{1}{2}\kappa(\theta - \phi)^2$ is stored in the spring.

Based on the second order development of x_1, y_1, x_2, y_2 an approximate Lagrangian can be written in terms of θ and φ.

More convenient variables can then be introduced such as

$$\begin{cases} x = l_1 \theta - l_2 \phi \\ z = l_2 \theta + l_1 \phi \end{cases}$$

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BHETSA Meeting
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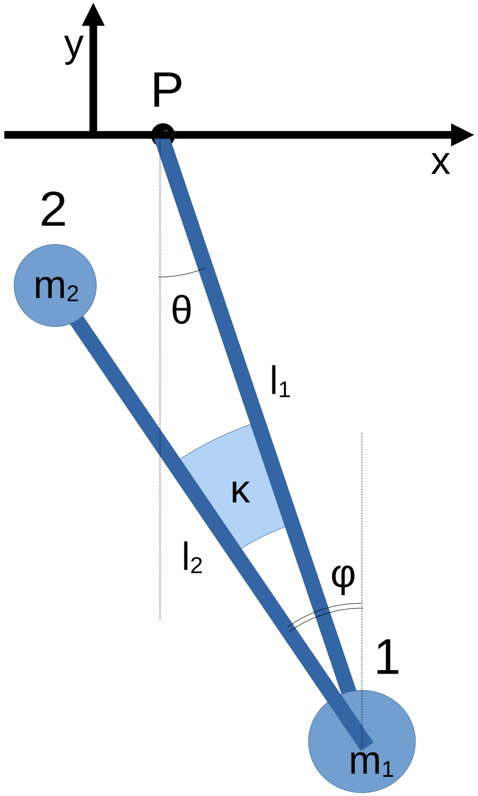


In terms of the variables $\begin{cases} x = l_1\theta - l_2\phi \\ z = l_2\theta + l_1\phi \end{cases}$

the Lagrange's equations of the system can be summarized as

$$\mathbf{M}\begin{pmatrix}\ddot{x}\\ \ddot{z}\end{pmatrix} = -\frac{g}{l_2}\mathbf{K}\begin{pmatrix}x\\ z\end{pmatrix} - \mathbf{N}\,\ddot{x}_P$$

where the constant matrices **M**, **K** and **N** are, respectively,

$$\mathbf{M} = \begin{pmatrix}\mu\lambda^4 + (\lambda^2+1)^2 & \mu\lambda^3 \\ \mu\lambda^3 & \mu\lambda^2\end{pmatrix}$$

$$\mathbf{K} = \begin{pmatrix}(\mu+1)\lambda^3 - 1 + \gamma(\lambda^2+1)^2 & (\mu+1)\lambda^2 + \lambda - \gamma(\lambda^2-1) \\ (\mu+1)\lambda^2 + \lambda - \gamma(\lambda^2-1) & (\mu+1)\lambda - \lambda^2 + \gamma(\lambda-1)^2\end{pmatrix}$$

$$\mathbf{N} = (\lambda^2+1)\begin{pmatrix}(\mu+1)\lambda^2 + 1 \\ \mu\lambda\end{pmatrix}$$

and some dimensionless parameters have been introduced:

$$\mu = \frac{m_1}{m_2}\ , \qquad \lambda = \frac{l_1}{l_2}\ , \qquad \gamma = \frac{\kappa}{m_2 g l_2}\ .$$

Oscillations

The equation describes a forced 2-DoF oscillator. Here we focus on the free oscillations (although the complete transfer function is available for further study).

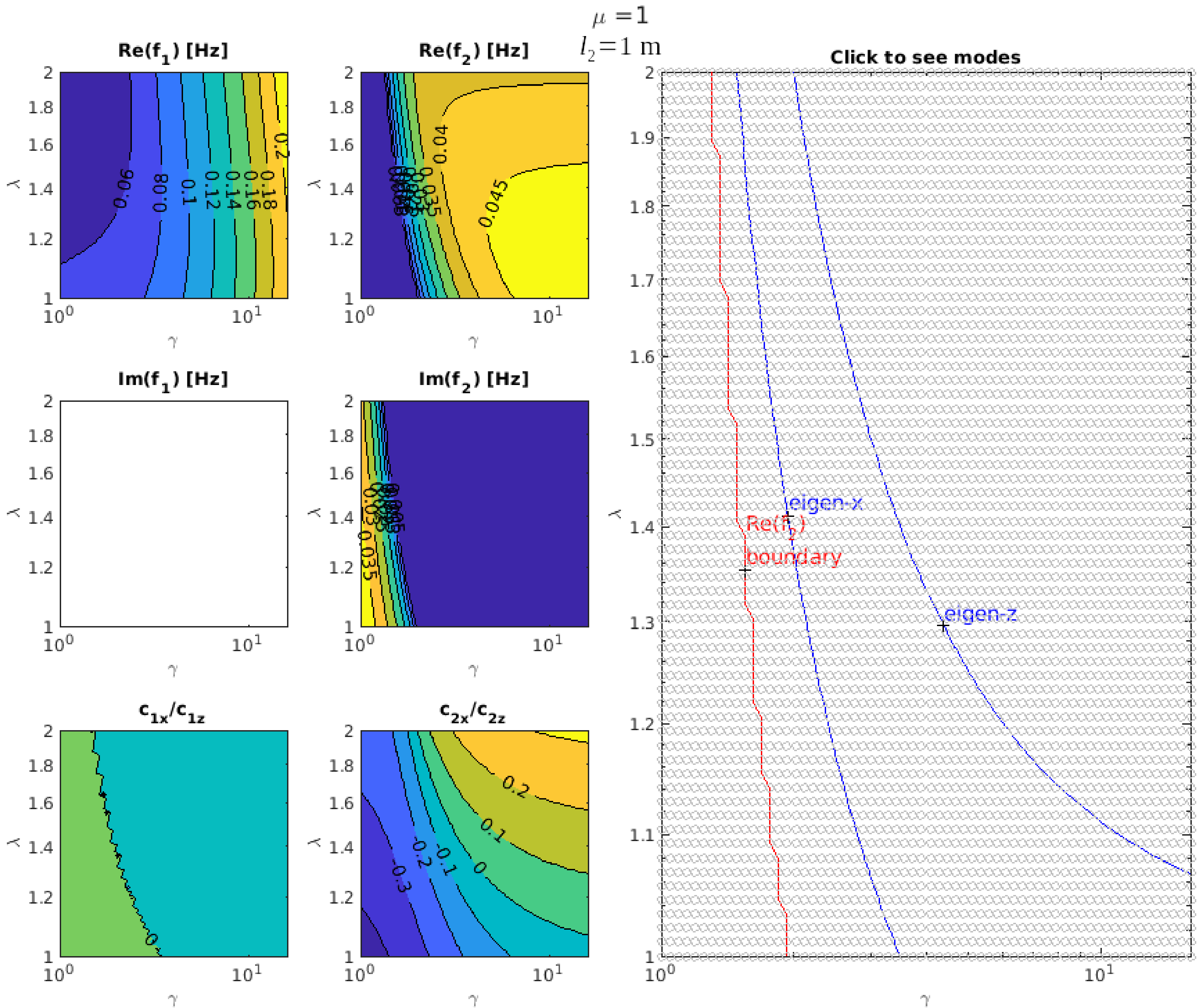
$$\mathbf{M}\begin{pmatrix}\ddot{x}\\ \ddot{z}\end{pmatrix} = -\frac{g}{l_2}\mathbf{K}\begin{pmatrix}x\\ z\end{pmatrix} - \mathbf{N}\ddot{x}_P \quad \Rightarrow \quad \begin{pmatrix}\ddot{x}\\ \ddot{z}\end{pmatrix} = -\frac{g}{l_2}\mathbf{M}^{-1}\mathbf{K}\begin{pmatrix}x\\ z\end{pmatrix}$$

The Laplace transform of the equation gives

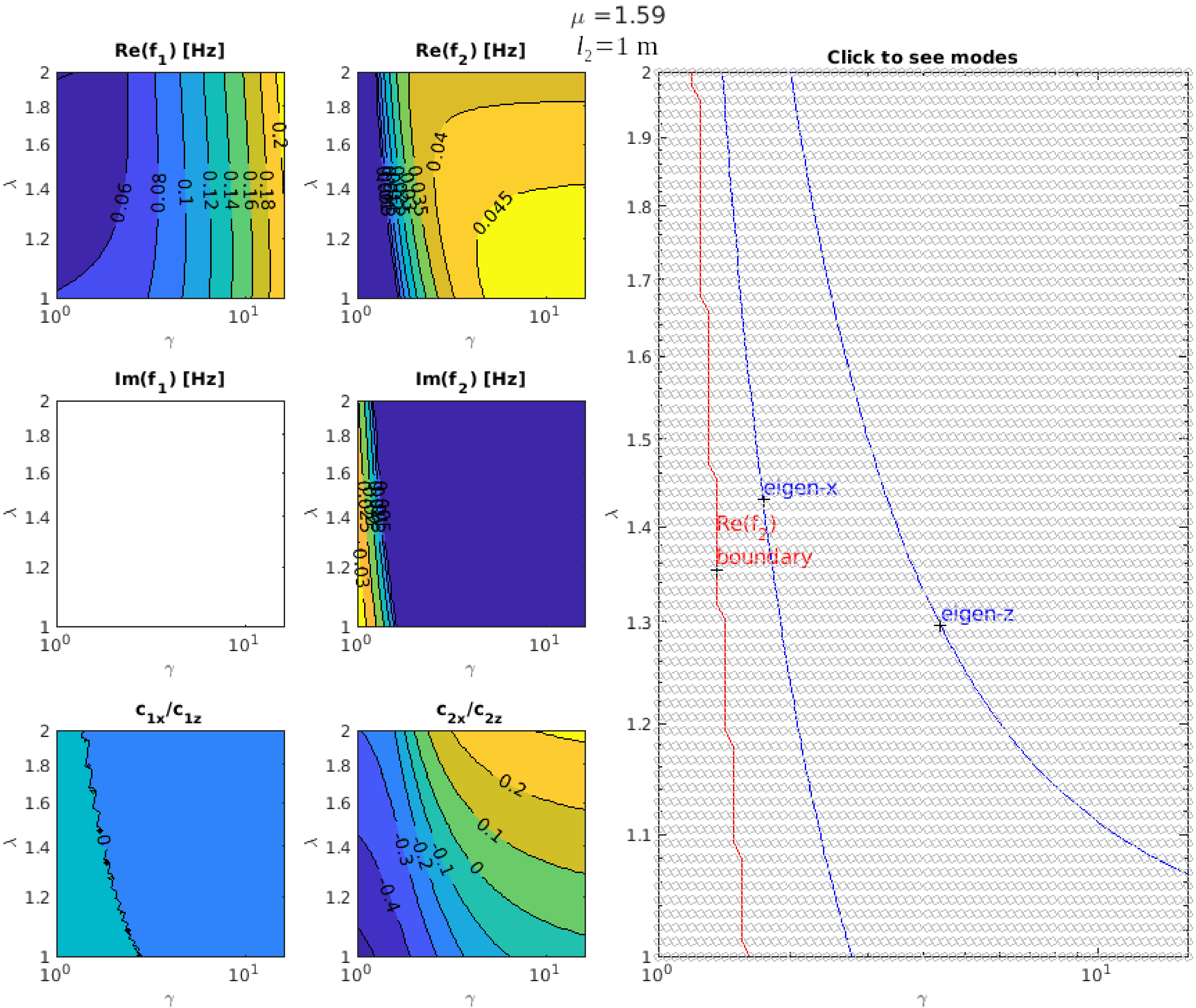
$$s^2\begin{pmatrix}\tilde{x}\\ \tilde{z}\end{pmatrix} = -\frac{g}{l_2}\mathbf{M}^{-1}\mathbf{K}\begin{pmatrix}\tilde{x}\\ \tilde{z}\end{pmatrix} \quad \text{or} \quad \left(s^2 + \frac{g}{l_2}\mathbf{M}^{-1}\mathbf{K}\right)\begin{pmatrix}\tilde{x}\\ \tilde{z}\end{pmatrix} = 0.$$

By putting the $\mathbf{M}^{-1}\mathbf{K}$ matrix in the diagonal form, one finds the natural frequencies of the system and the eigen-states of the oscillator that are presented hereafter.

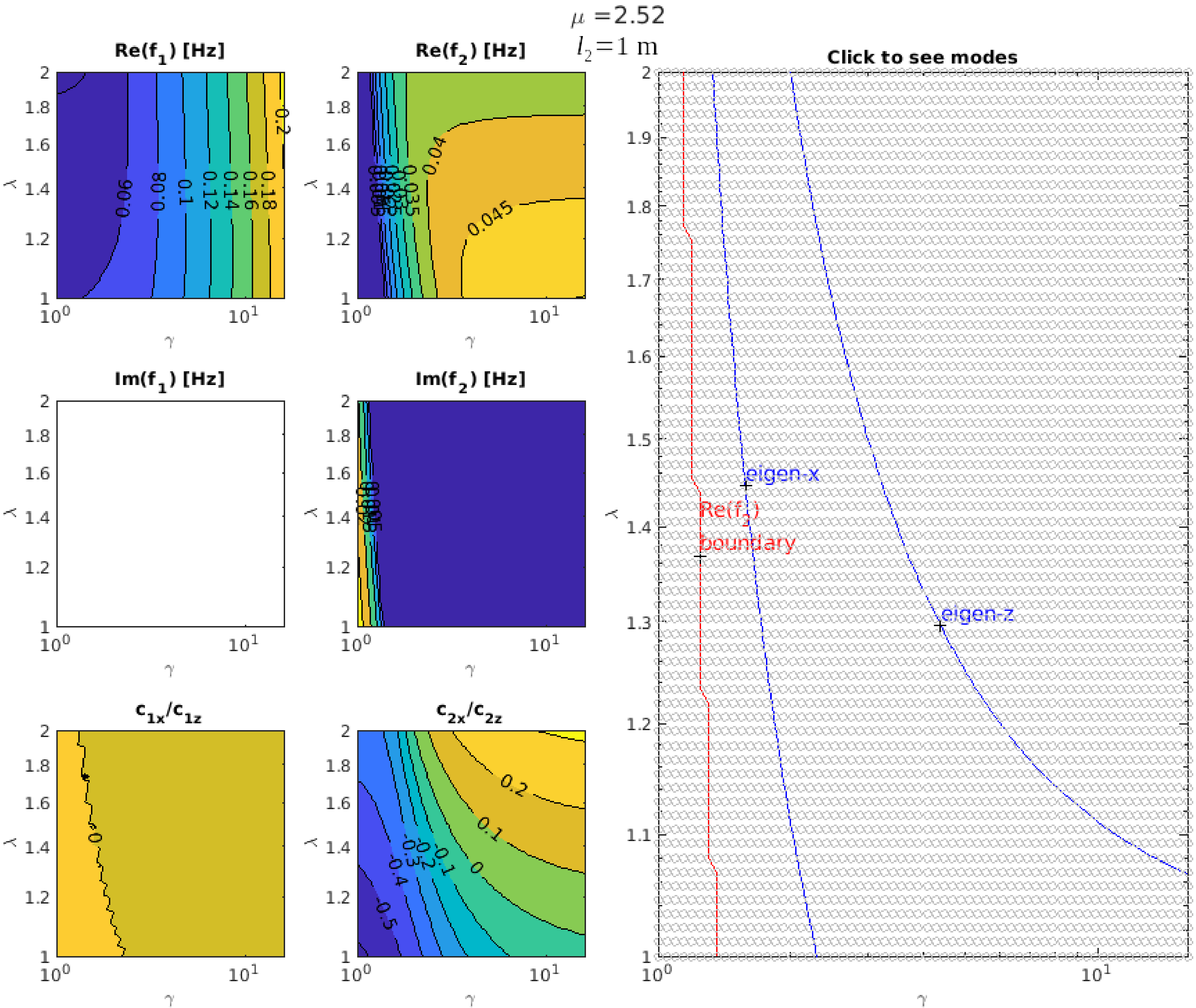
All plots are worked out with $l_2 = 1\ \mathrm{m}$. All **frequencies scale as** $\frac{1}{\sqrt{l_2}}$.



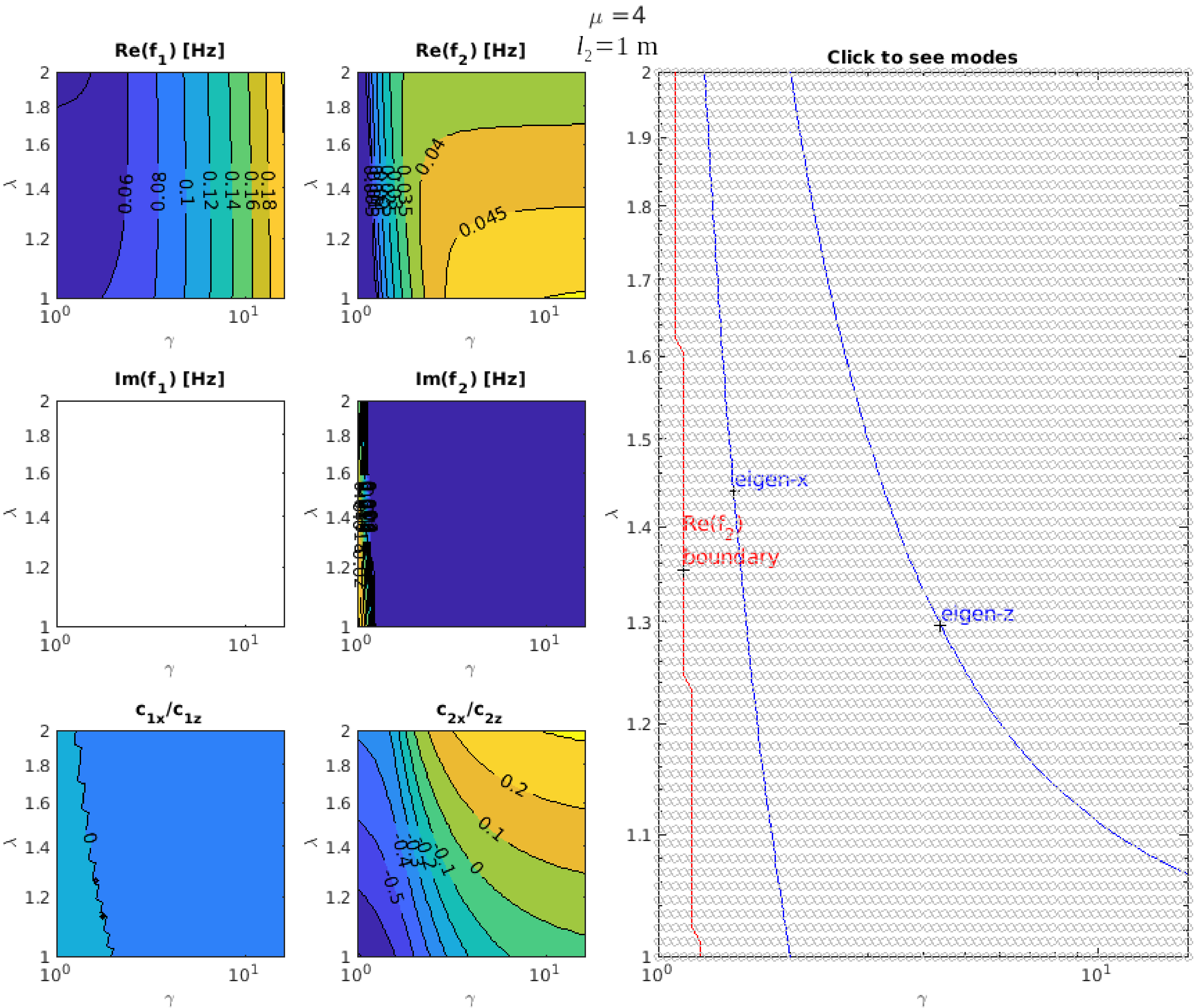
μ = 1
l_2 = 1 m
Re(f1) [Hz]
Re(f2) [Hz]
Im(f1) [Hz]
Im(f2) [Hz]
c1x/c1z
c2x/c2z
Click to see modes
γ
λ
eigen-x
eigen-z
Re(f2)
boundary



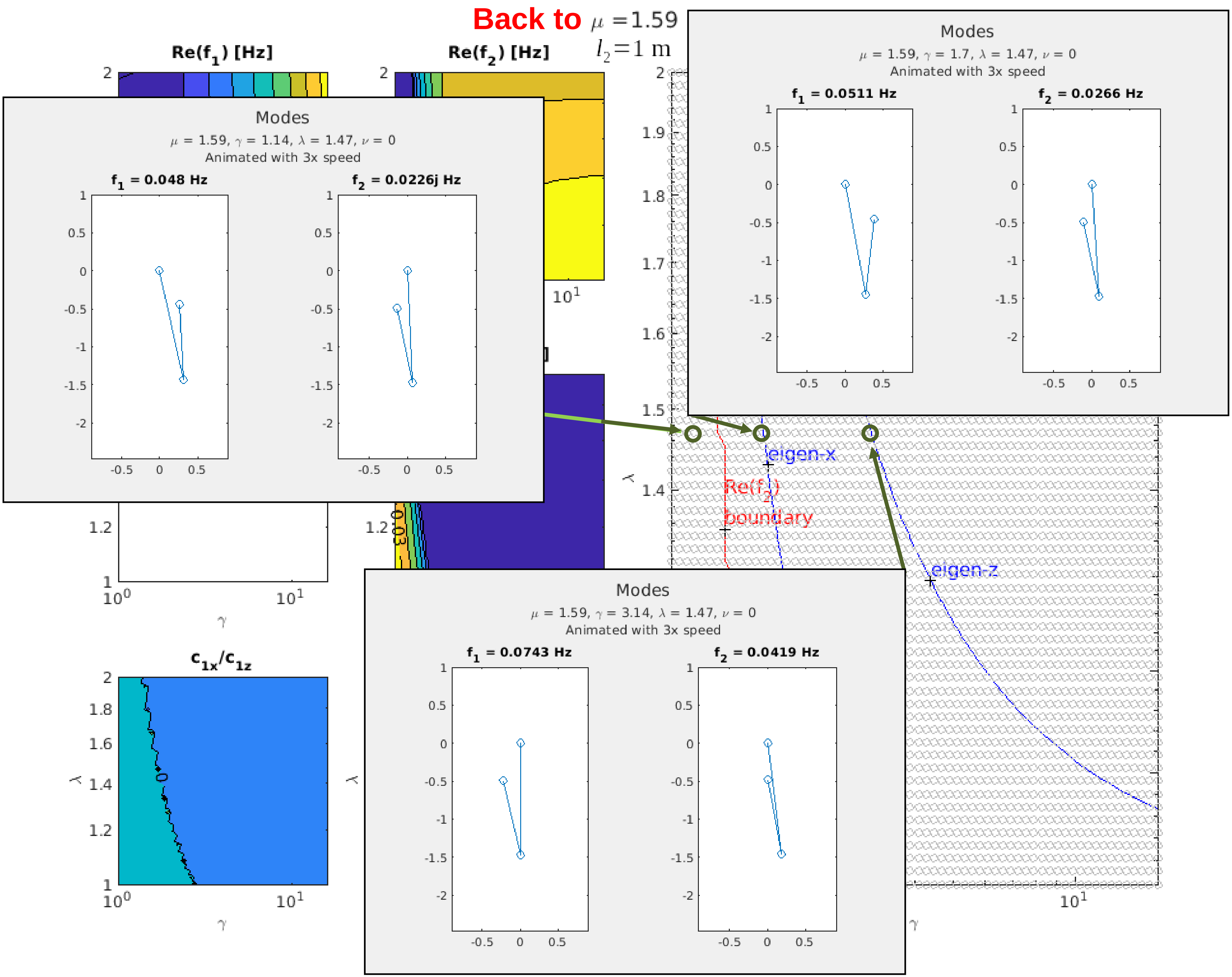
μ =1.59
l_2=1 m
Re(f_1) [Hz]
Re(f_2) [Hz]
Im(f_1) [Hz]
Im(f_2) [Hz]
c_{1x}/c_{1z}
c_{2x}/c_{2z}
Click to see modes
γ
λ
eigen-x
eigen-z
Re(f_2) boundary



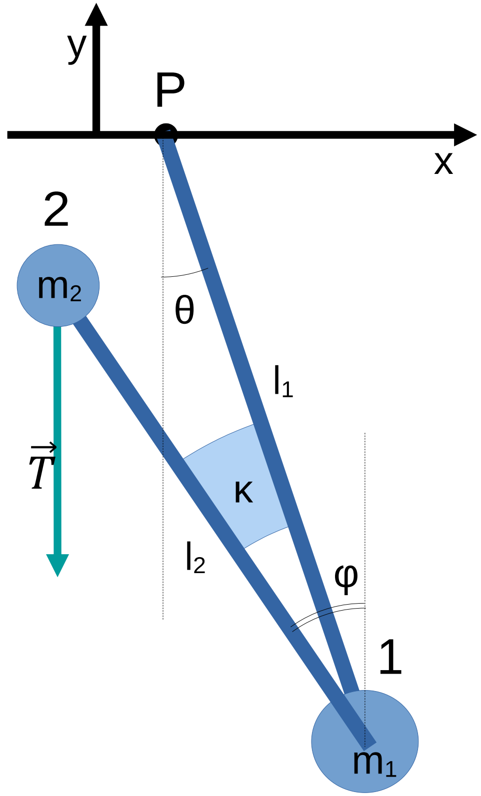
μ =2.52
l2=1 m
Re(f1) [Hz]
Re(f2) [Hz]
Click to see modes
Im(f1) [Hz]
Im(f2) [Hz]
c1x/c1z
c2x/c2z
eigen-x
eigen-z
Re(f1) boundary
γ
λ



μ =4
l2=1 m
Re(f1) [Hz]
Re(f2) [Hz]
Click to see modes
Im(f1) [Hz]
Im(f2) [Hz]
c1x/c1z
c2x/c2z
λ
γ
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.04
0.045
0.035
0.2
0.1
0
-0.5
eigen-x
eigen-z
Re(f2)
boundary


Back to $\mu = 1.59$
$l_2 = 1$ m
Re(f_1) [Hz]
Re(f_2) [Hz]
Modes
μ = 1.59, γ = 1.14, λ = 1.47, ν = 0
Animated with 3x speed
f_1 = 0.048 Hz
f_2 = 0.0226j Hz
Modes
μ = 1.59, γ = 1.7, λ = 1.47, ν = 0
Animated with 3x speed
f_1 = 0.0511 Hz
f_2 = 0.0266 Hz
eigen-x
Re(f_2) boundary
eigen-z
Modes
μ = 1.59, γ = 3.14, λ = 1.47, ν = 0
Animated with 3x speed
f_1 = 0.0743 Hz
f_2 = 0.0419 Hz
c_{1x}/c_{1z}
γ
λ

The Pendulum-Inverted Pendulum oscillator with a constant vertical load



Let's suppose that a load is suspended to the device and that this load can be approximated with a constant vertical force.

The Lagrange's equations of the system become

$$\mathbf{M}\begin{pmatrix}\ddot{x}\\ \ddot{z}\end{pmatrix}=-\frac{g}{l_2}(\mathbf{K}+\mathbf{K}_T)\begin{pmatrix}x\\ z\end{pmatrix}-\mathbf{N}\ddot{x}_P$$

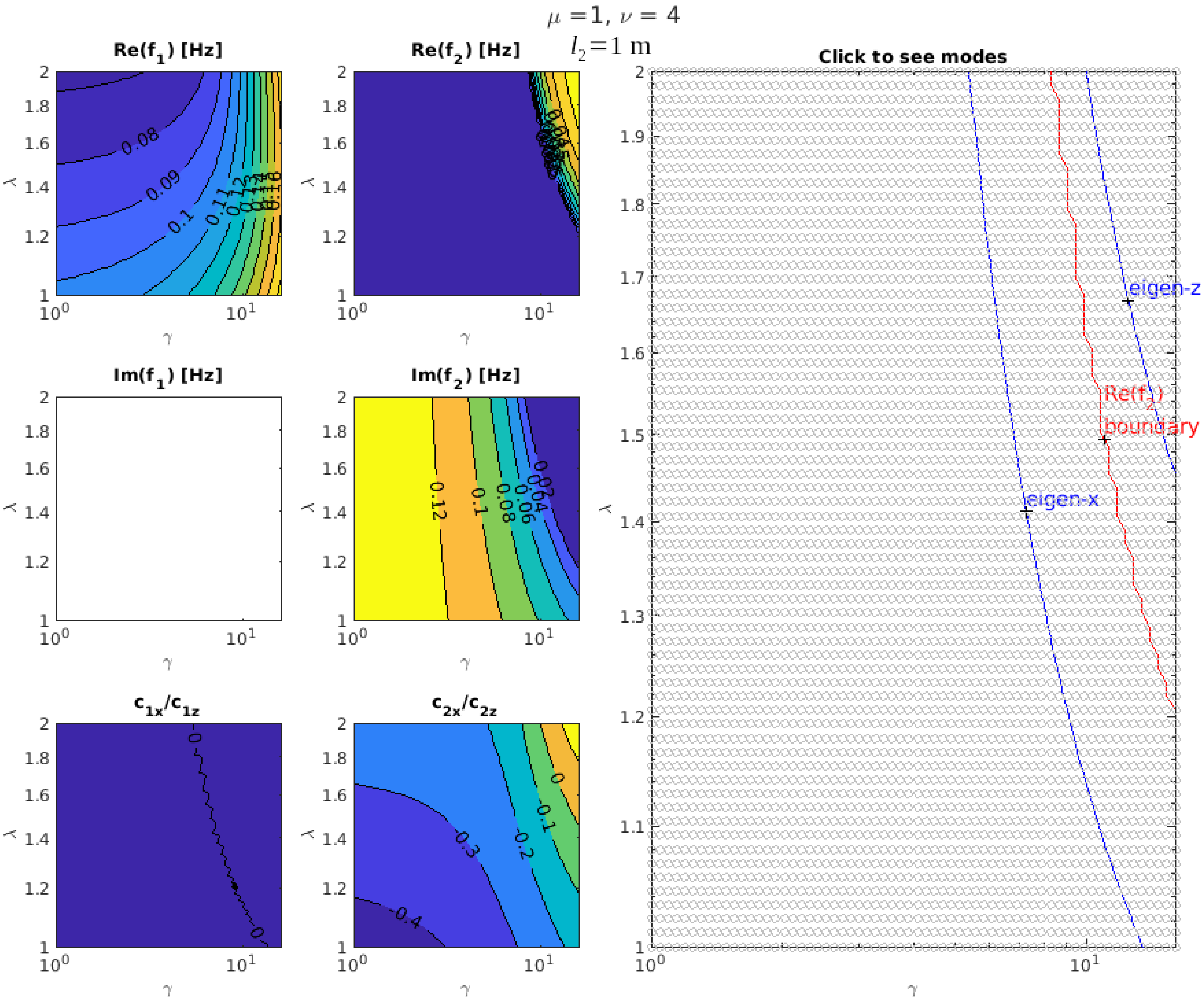
where a new constant matrix $\mathbf{K}_T$ is introduced

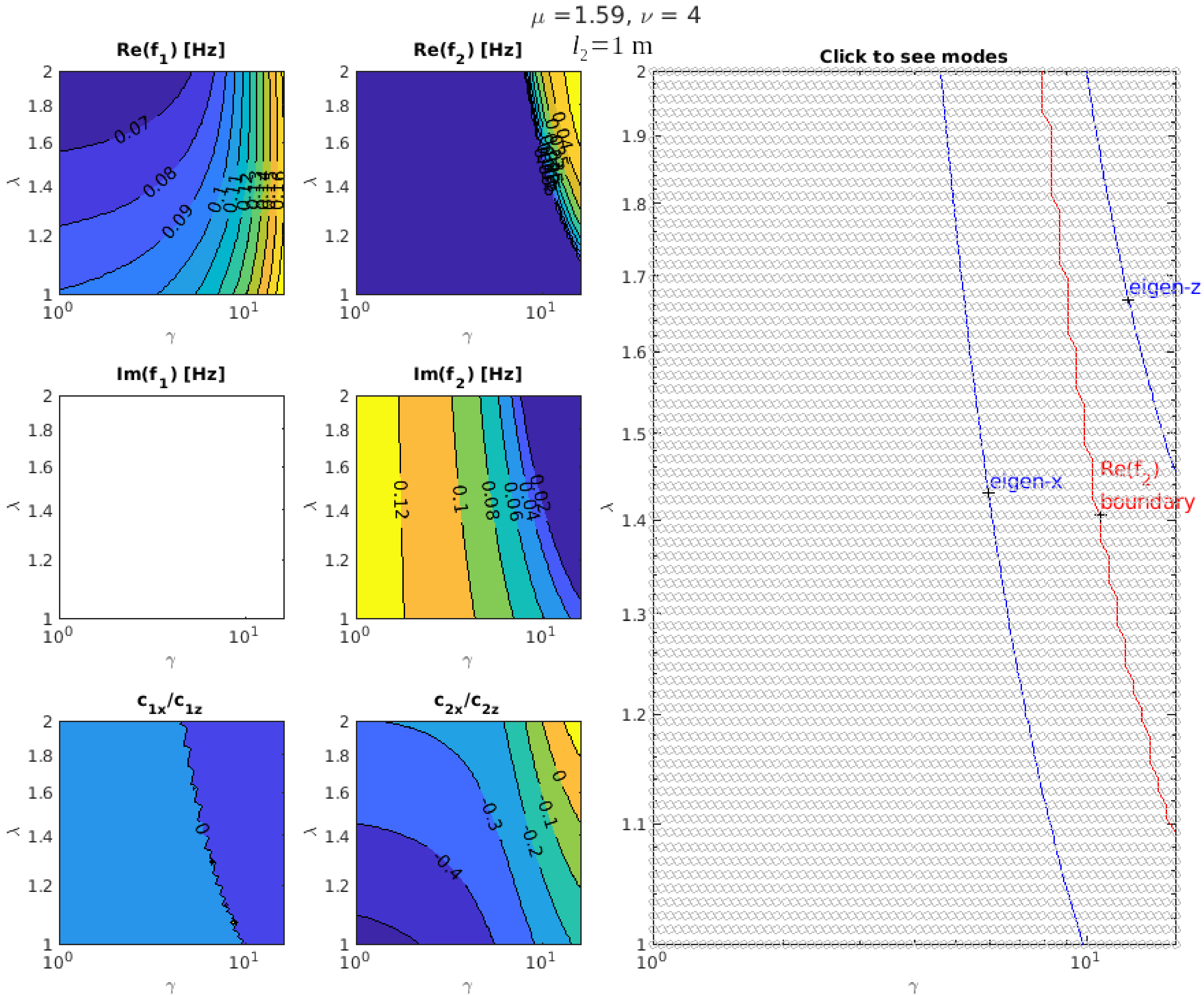
$$\mathbf{K}_T=\nu\begin{pmatrix}\lambda^3-1 & \lambda(\lambda+1)\\ \lambda(\lambda+1) & \lambda(1-\lambda)\end{pmatrix}$$

along with the new dimensionless parameter $\nu=\dfrac{T}{m_2 g}$.

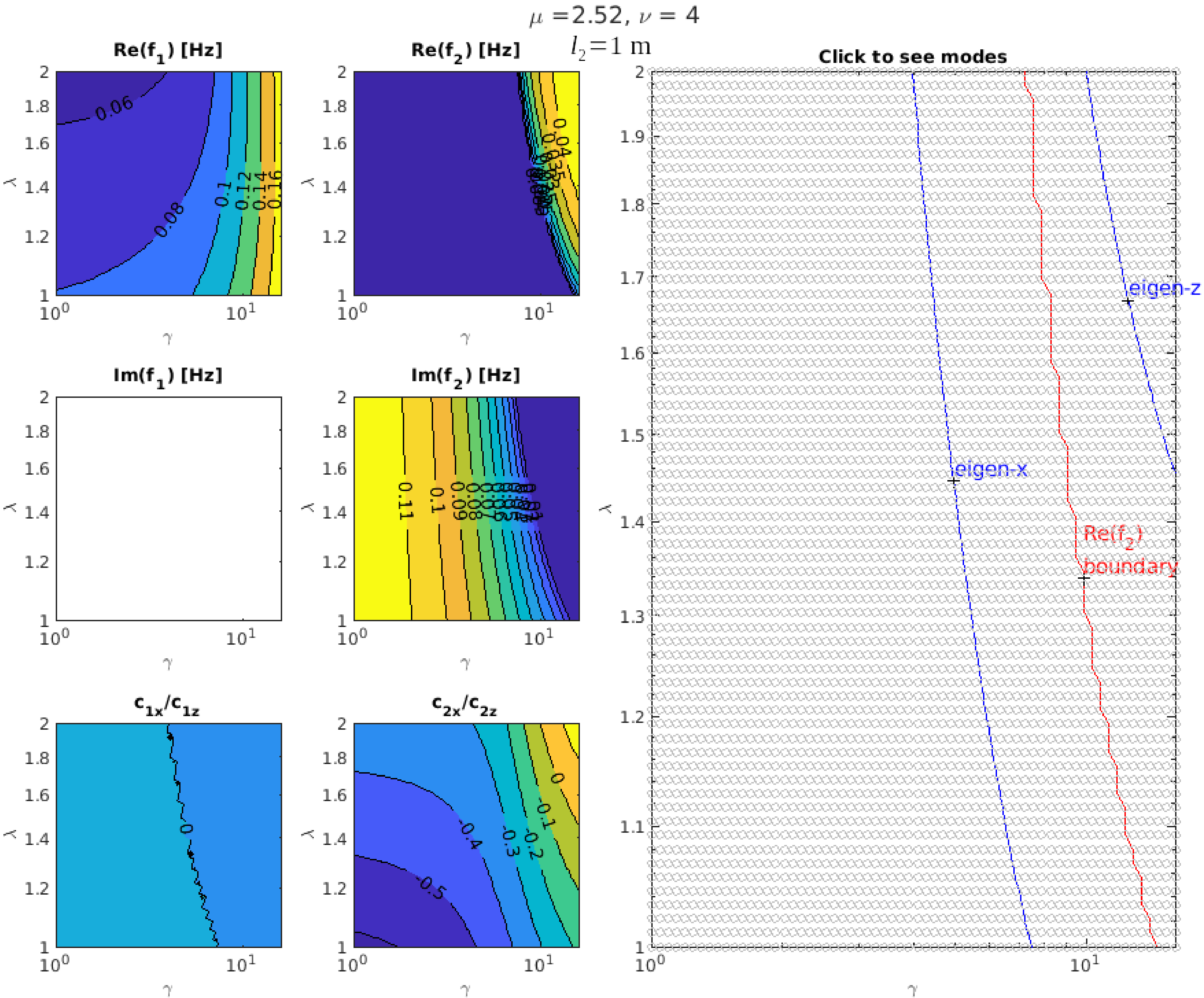
The natural modes of the loaded system can thus be found by solving

$$\left[s^2+\frac{g}{l_2}\mathbf{M}^{-1}(\mathbf{K}+\mathbf{K}_T)\right]\begin{pmatrix}\tilde{x}\\ \tilde{z}\end{pmatrix}=0$$

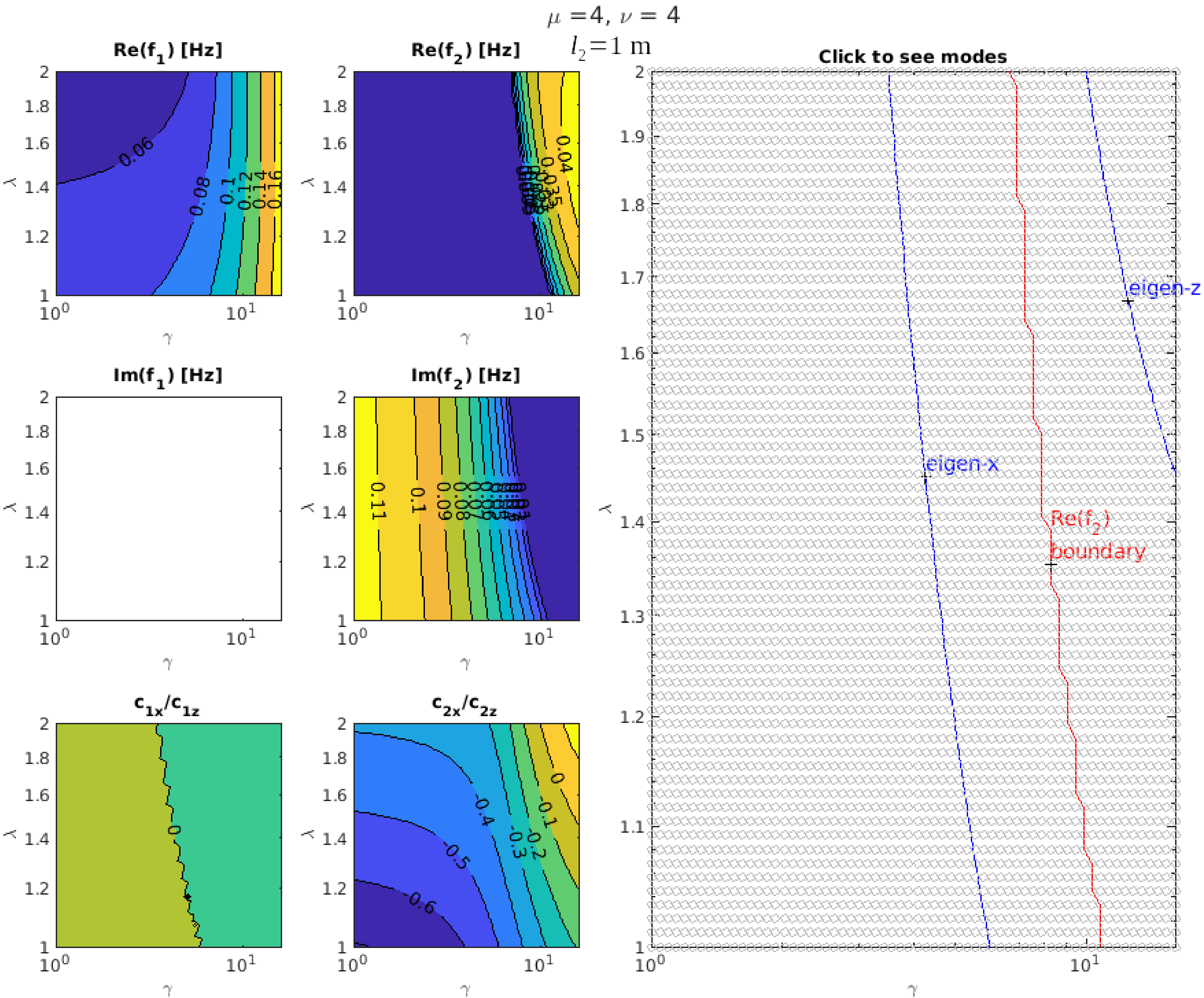

μ =1, ν = 4
l2=1 m
Re(f1) [Hz]
Re(f2) [Hz]
Im(f1) [Hz]
Im(f2) [Hz]
c1x/c1z
c2x/c2z
Click to see modes
eigen-z
eigen-x
Re(f2) boundary
γ
λ



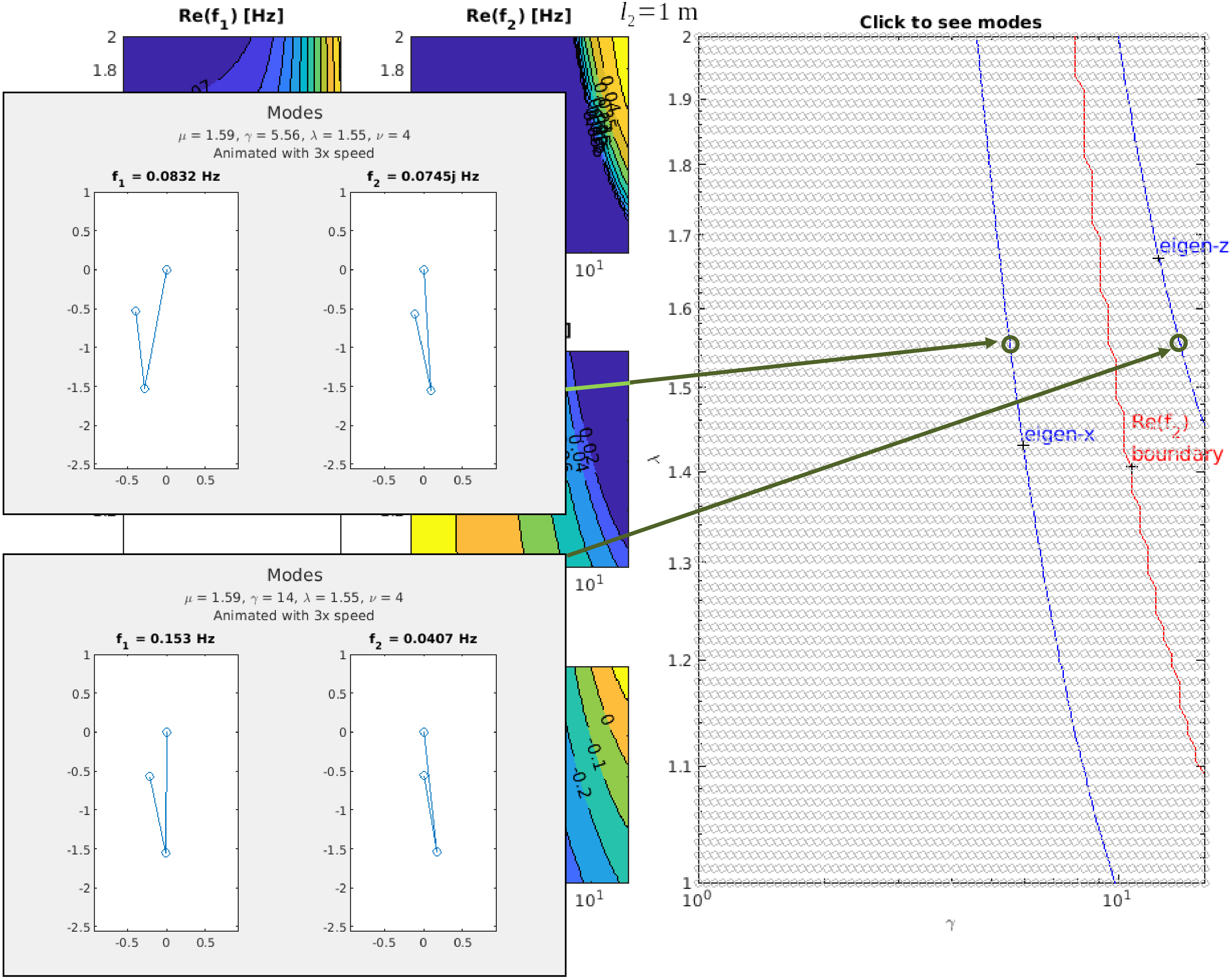
μ =1.59, ν = 4
l_2=1 m
Re(f_1) [Hz]
Re(f_2) [Hz]
Im(f_1) [Hz]
Im(f_2) [Hz]
c_1x/c_1z
c_2x/c_2z
Click to see modes
eigen-z
eigen-x
Re(f_2) boundary
λ
γ



μ =2.52, ν = 4
l_2=1 m
Re(f1) [Hz]
Re(f2) [Hz]
Click to see modes
Im(f1) [Hz]
Im(f2) [Hz]
c1x/c1z
c2x/c2z
eigen-z
eigen-x
Re(f2) boundary
λ
γ


μ =4, ν = 4
l2=1 m
Re(f1) [Hz]
Re(f2) [Hz]
Click to see modes
Im(f1) [Hz]
Im(f2) [Hz]
c1x/c1z
c2x/c2z
eigen-z
eigen-x
Re(f2)
boundary
λ
γ

Back to $\mu = 1.59$, $\nu = 4$

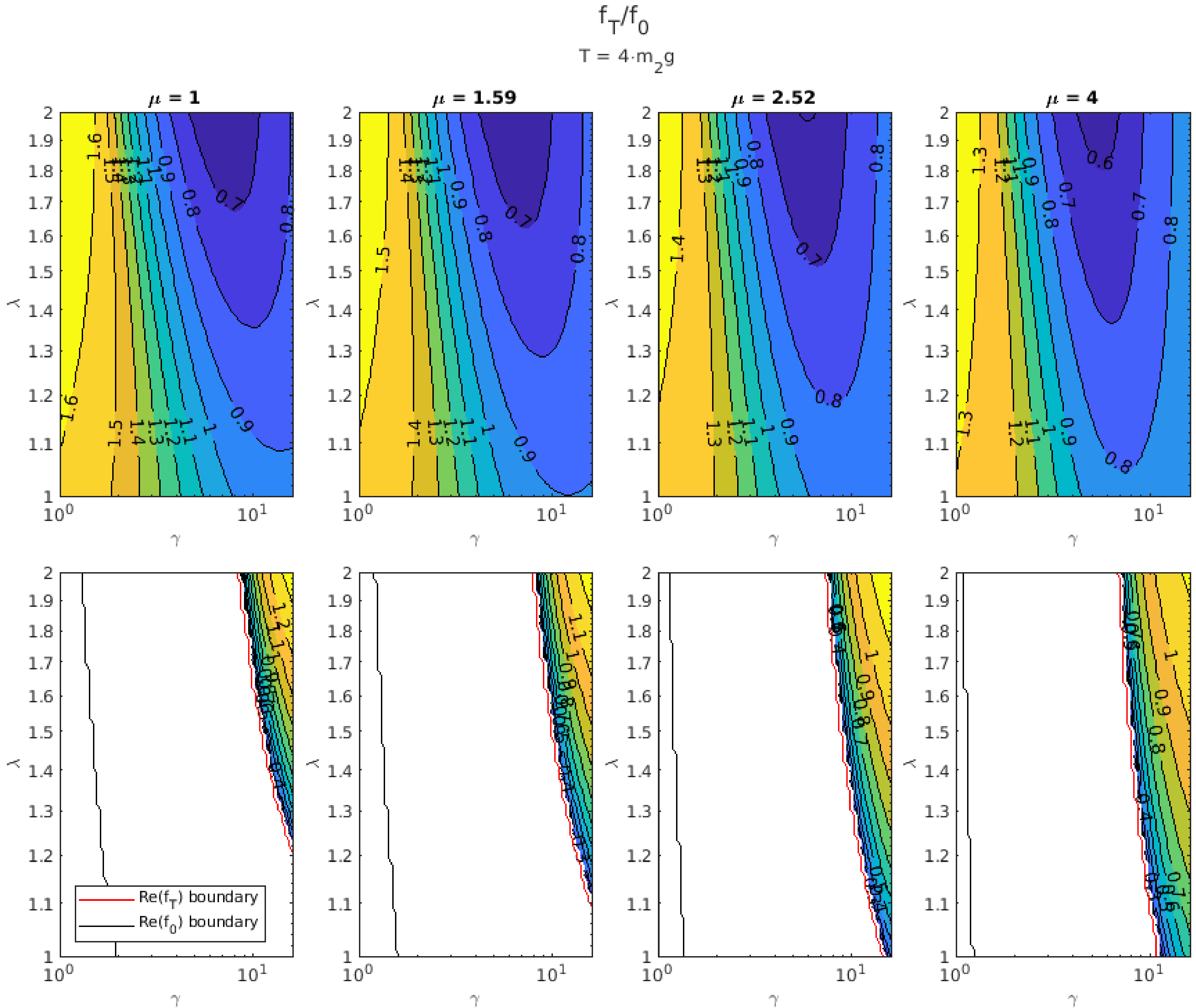


Variable definitions and values

System variables: $\begin{cases} x = l_1\theta - l_2\phi \\ z = l_2\theta + l_1\phi \end{cases}$

Parameter definitions:	$\mu = \frac{m_1}{m_2}$	$\lambda = \frac{l_1}{l_2}$	$\gamma = \frac{\kappa}{m_2 g l_2}$	$\nu = \frac{T}{m_2 g}$
Interval:	**1** to **4**	**1** to **2**	**1** to **16**	**0** to **4**
No of values:	4	64	64	2
Equally spaced in:	Log scale	Log scale	Log scale	

All plots are worked out with $l_2 = 1\ \mathrm{m}$. All frequencies scale as $\frac{1}{\sqrt{l_2}}$



f_T/f_0
T = 4·m$_2$g
μ = 1
μ = 1.59
μ = 2.52
μ = 4
λ
γ
Re(f_T) boundary
Re(f_0) boundary